FRANCIS S. TSE
University of Cincinnati
IVAN E. MORSE
University of Cincinnati
ROLLAND T. HINKLE
Michigan State University

Mechanical Vibrations
Theory and Applications

SECOND EDITION

Allyn and Bacon, Inc.
Boston / London / Sydney / Toronto

Copyright © 1978, 1%3 by Allyn and Bacon, Inc.
470 Atlantic Avenue, Boston, Massachusetts 02210.

All rights reserved. Printed in the United States of America.
No part of the material protected by this copyright notice
may be reproduced or utilized in any form or by any means,
electronic or mechanical, including photocopying, recording,
or by any information storage and retrieval system, without
written permission from the copyright owner.

10 9 8 7 6 5 85 84 83

Library of Congress Cataloging in Publication Data

Tse, Francis Sing
Mechanical vibrations.

(Allyn and Bacon series in Mechanical engineering
and applied mechanics)
Includes index.
1. Vibrations. I. Morse, Ivan E., joint
author. II. Hinkle, Rolland Theodore, joint
author. III. Title.
TA355.T77 1978 620.3 77-20933

ISBN 0-205-05940-6
ISBN 0-205-06670-4 (International)

Contents

Preface

Vibration is the study of oscillatory motions. The ultimate goals of this study are to determine the effect of vibration on the performance and safety of systems, and to control its effects. With the advent of high performance machines and environmental control, this study has become a part of most engineering curricula.

This text presents the fundamentals and applications of vibration theory. It is intended for students taking either a first course or a one-year sequence in the subject at the junior or senior level. The student is assumed to have an elementary knowledge of dynamics, strength of materials, and differential equations, although summaries of several topics are included in the appendices for review purposes. The format of its predecessor is retained, but the text material has been substantially rewritten. In view of the widespread adoption of the International System of Units (SI) by the industrial world, SI units are used in the problems.

The objectives of the text are first, to establish a sense of engineering reality, second, to provide adequate basic theory, and finally, to generalize these concepts for wider applications: The primary focus of the text is on the engineering significance of the physical quantities, with the mathematical structure providing a supporting role. Throughout the text, examples of applications are given before the generalization to give the student a frame of reference, and to avoid the pitfall of overgeneralization. To further enhance engineering reality, detailed digital computations for discrete systems are presented so that the student can solve meaningful numerical problems.

The first three chapters examine systems with one degree of freedom. General concepts of vibration are described in Chapter 1. The theory of time and frequency domain analysis is introduced in Chapter **2** through the study of a generalized model, consisting of the mass, spring, damper, and excitation elements. This provides the basis for modal analyses in subsequent chapters. The applications in Chapter **3** demonstrate that the elements of the model are, in effect, equivalent quantities. Although the same theory is used, the appearance of a system in an engineering problem may differ greatly from that of the model. The emphasis of Chapter **3** is on problem formulation. Through the generalization and classification of problems in the chapter, a new encounter will not appear as a stranger.

Discrete systems are introduced in Chapter 4 using systems with two degrees of freedom. Coordinate coupling is treated in detail. Common methods of finding natural frequencies are described in Chapter 5. The material in these chapters is further developed in Chapter 6 using matrix techniques and relating the matrices to energy quantities. Thus, the student would not feel the artificiality in the numerous coordinate transformations in the study.

The one-dimensional wave equation and beam equation of continuous systems are discussed in Chapter 7. The material is organized to show the similarities between continuous and discrete systems. Chapter 8, on nonlinear systems, explains certain common phenomena that cannot be predicted by linear theory. The chapter consists of two main parts, conforming to the geometric and analytical approaches to nonlinear studies.

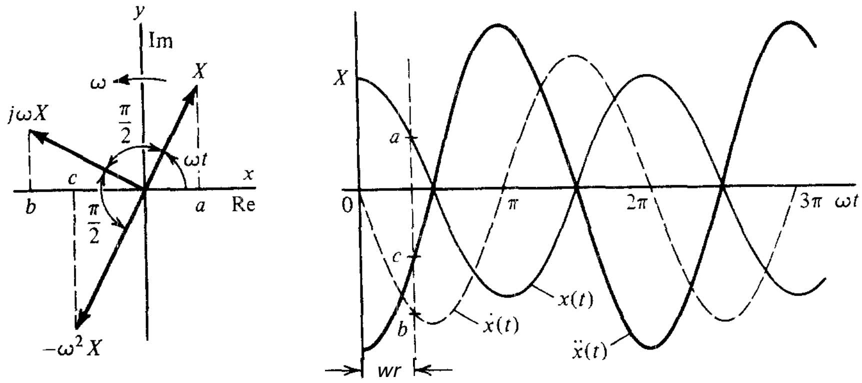
The digital computation in Chapter 9 is organized to follow the sequence of topics presented in the prior chapters and can be assigned concurrently with the text material. The programs listed in Table 9–1 are sufficient for the computation and plotting of results for either damped or undamped discrete systems. Detailed explanations are given to aid the student in executing the programs. The programs are almost conversational and only a minimal knowledge of FORTRAN is necessary for their execution.

The first five chapters constitute the core of an elementary, one-quarter terminal course at the junior level. Depending on the purpose of the particular course, parts of Section 3–5 can be used as assigned reading. Sections **3–6** through 3–8, Section 4–9, and Sections 5–4 through 5–6 may be omitted without loss of continuity.

For a one-semester senior or dual-level course, the instructor may wish to use Chapters 1 through 4, Chapter 6, and portions of Chapter 7 or **8.** Some topics, such as equivalent viscous damping, may be omitted.

Alternatively, the text has sufficient material for a one-year sequence at the junior or senior level. Generally, the first course in mechanical vibrations is required and the second is an elective. The material covered will give the student **a** good background for more advanced studies.

We would like to acknowledge our indebtedness to many friends, students, and colleagues for their suggestions, to the numerous writers **who**



(a) Vectorial representation (b) Harmonic motions

FIG. 1-8. *Displacement, velocity, and acceleration vectors.*

Thus, each differentiation is equivalent to the multiplication of the vector by $j\omega$. Since X is the magnitude of the vector **X**, ω is real, and $|j|=1$, each differentiation changes the magnitude by a factor of ω. Since the multiplication of a vector by j is equivalent to advancing it by a phase angle of 90°, each differentiation also advances a vector by 90".

If a given harmonic displacement is $x(t)=X\cos\omega t$, the relations between the displacement and its velocity and acceleration are

$$
\begin{aligned}
\text{Displacement } x &= \text{Re}[Xe^{j\omega t}] = X\cos\omega t \\
\text{Velocity } \dot{x} &= \text{Re}[j\omega X e^{j\omega t}] = -\omega X\sin\omega t \\
&= \omega X\cos(\omega t+90°) \\
\text{Acceleration } \ddot{x} &= \text{Re}[(j\omega)^2 X e^{j\omega t}] = -\omega^2 X\cos\omega t \\
&= \omega^2 X\cos(\omega t+180°)
\end{aligned}
\tag{1-11}
$$

These relations are identical to those shown in Eqs. (1-4) to (1-6). The representation of displacement, velocity, and acceleration by rotating vectors is illustrated in Fig. 1-8. Since the given displacement $x(t)$ is a cosine function, or along the real axis, the velocity and acceleration must be along the real axis. Hence the real parts of the respective vectors give the physical quantities at the given time t.

Harmonic functions can be added graphically be means of vector addition. The vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ representing the motions $X_1\cos$ wt and $X_2\cos(\omega t+a)$, respectively, are added graphically as shown in Fig. 1-9(*a*). The resultant vector **X** has a magnitude

$$X=\sqrt{(X_1+X_2\cos\alpha)^2+(X_2\sin\alpha)^2}$$

and a phase angle

$$\beta=\tan^{-1}\frac{X_2\sin\alpha}{X,+X_2\cos\alpha}$$

1

Introduction

1-1 PRIMARY OBJECTIVE

The subject of vibration deals with the oscillatory motion of **dynamic systems.** A dynamic system is a combination of matter which possesses mass and whose parts are capable of relative motion. All bodies possessing mass and elasticity are capable of vibration. The mass is inherent of the body, and the elasticity is due to the **relative** motion of the parts of the body. The system considered may be very simple or complex. It may be in the form of a structure, a machine or its components, or a group of machines. The oscillatory motion of the system may be objectionable, trivial, or necessary for performing a task.

The objective of the designer is to control the vibration when it is objectionable and to enhance the vibration when it **is** useful, although vibrations in general are undesirable. Objectionable vibrations in a machine **may** cause the loosening of parts, its malfunctioning, or its eventual failure. On the other hand, shakers in foundries and vibrators in testing machines require vibration. The performance of many instruments depends on the proper control of the vibrational characteristics of the devices.

The primary objective of our study is to analyze the oscillatory motion of dynamic systems and the forces associated with the motion. The ultimate goal in the study of vibration is to determine its effect on the performance and safety of the system under consideration. The analysis of the oscillatory motion is an important step towards this goal.

Our study begins with the description of the elements in a vibratory system, the introduction of some terminology and concepts, and the discussion of simple harmonic motion. These will be used throughout the text. Other concepts and terminology will be introduced in the appropriate places as needed.

1-2 ELEMENTS OF A VIBRATORY SYSTEM

The elements that constitute a vibratory system are illustrated in Fig. 1-1. They are idealized and called (1) the mass, (2) the spring, (3) the damper, and (4) the excitation. The first three elements describe the physical system. For example, it can be said that a given system consists of a mass, a spring, and a damper arranged as shown in the figure. Energy may be stored in the mass and the spring and dissipated in the damper in the form of heat. Energy enters the system through the application of an excitation. As shown in Fig. 1-1, an excitation force is applied to the mass m of the system.

The mass m is assumed to be a rigid body. It executes the vibrations and can gain or lose kinetic energy in accordance with the velocity change of the body. From Newton's law of motion, the product of the mass and its acceleration is equal to the force applied *to* the mass, and the acceleration takes place in the direction in which the force acts. Work is force times displacement in the direction of the force. The work is transformed into the kinetic energy of the mass. The kinetic energy increases if work is positive and decreases if work is negative.

The spring k possesses elasticity and is assumed to be of negligible mass. A spring force exists if the spring is deformed, such as the extension or the compression of a coil spring. Therefore the spring force exists only if there is a relative displacement between the two ends of the spring. The work done in deforming a spring is transformed into potential energy, that is, the strain energy stored in the spring. A linear spring is one that obeys Hooke's law, that is, the spring force is proportional to the spring deformation. The constant of proportionality, measured in force per unit deformation, is called the stiffness, or the spring constant k.

The damper c has neither mass nor elasticity. Damping force exists only if there is relative motion between the two ends of the damper. The work or the energy input to a damper is converted into heat. Hence the damping element is nonconservatiue. Viscous damping, in which the damping force is proportional to the velocity, is called linear damping. Viscous damping, or its equivalent, is generally assumed in engineering.

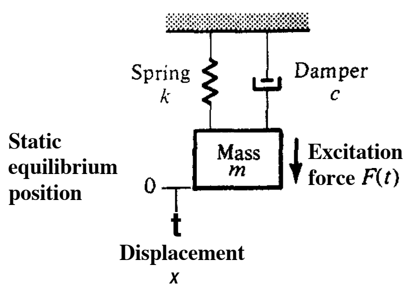


FIG. 1-1. *Elements of a vibratory system.*

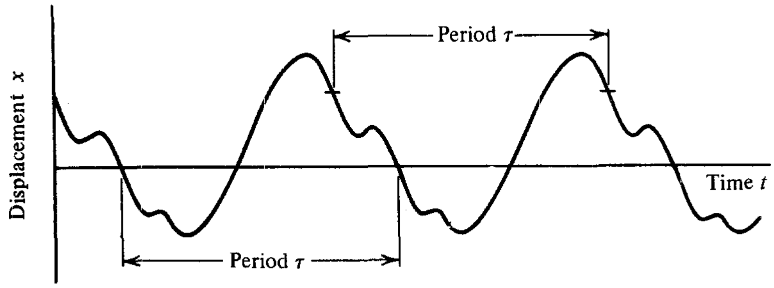


FIG. 1-2. A *periodic motion.*

The viscous damping ***coefficient*** c is measured in force per unit velocity. Many types of nonlinear damping are commonly encountered. For example, the frictional drag of a body moving in a fluid is approximately proportional to the velocity squared, but the exact value of the exponent is dependent on many variables.

Energy enters a system through the application of an excitation. An excitation force may be applied to the mass and/or an excitation motion applied to the spring and the damper. An excitation force $F(t)$ applied to the mass m is illustrated in Fig. 1-1. The excitation varies in accordance with a prescribed function of time. Hence the excitation is always known at a given time. Alternatively, if the system is suspended from a support, excitation may be applied to the system through imparting a prescribed motion to the support. In machinery, excitation often arises from the unbalance of the moving components. The vibrations of dynamic systems under the influence of an excitation is called forced vibrations. Forced vibrations, however, are often defined as the vibrations that are caused and maintained by a periodic excitation.

If the vibratory motion is periodic, the system repeats its motion at equal time intervals as shown in Fig. 1-2. The minimum time required for the system to repeat its motion is called a period τ, which is the time to complete one cycle of motion. Frequency f is the number of times that the motion repeats itself per unit time. A motion that does not repeat itself at equal time intervals is called an aperiodic motion.

A dynamic system can be set into motion by some initial conditions, or disturbances at time equal to zero. If no disturbance or excitation is applied after the zero time, the oscillatory motions of the system are called free vibrations. Hence free vibrations describe the natural behavior or the natural modes of vibration of a system. The initial condition is an energy input. If a spring is deformed, the input is potential energy. If a mass is given an initial velocity, the input is kinetic energy. Hence initial conditions are due to the energy initially stored in the system.

If the system does not possess damping, there is no energy dissipation. Initial conditions would cause the system to vibrate and the free vibration of an undamped system will not diminish with time. If a system possesses damping, energy will be dissipated in the damper. Hence the free vibrations will eventually die out and the system then remain at its static equilibrium position. Since the energy stored is due to the initial conditions, free vibrations also describe the natural behavior of the system as it relaxes from the initial state to its static equilibrium.

For simplicity, lumped masses, linear springs, and viscous dampers will be assumed unless otherwise stated. Systems possessing these characteristics are called linear systems. An important property of linear systems is that they follow the principle of superposition. For example, the resultant motion of the system due to the simultaneous application of two excitations is a linear combination of the motions due to each of the excitations acting separately. The values of m, c, and k of the elements in Fig. 1-1 are often referred to as the system parameters. For a given problem, these values are assumed time invariant. Hence the coefficients or the parameters in the equations are constants. The equation of motion of the system becomes a linear ordinary differential equation with constant coefficients, which can be solved readily.

Note that the idealized elements in Fig. 1-1 form a model of a vibratory system which in reality can be quite complex. For example, a coil spring possesses both mass and elasticity. In order to consider it as an idealized spring, either its mass is assumed negligible or an appropriate portion of its mass is lumped together with the other masses of the system. The resultant model is a lumped-parameter, or discrete, system. For example, a beam has its mass and elasticity inseparably distributed along its length. The vibrational characteristics of a beam, or more generally of an elastic body or a continuous system, can be studied by this approach if the continuous system is approximated by a finite number of lumped parameters. This method is a practical approach to the study of some very complicated structures, such as an aircraft.

In spite of the limitations, the lumped-parameter approach to the study of vibration problems is well justified for the following reasons. (1) Many physical systems are essentially discrete systems. (2) The concepts can be extended to analyze the vibration of continuous systems. (3) Many physical systems are too complex to be investigated analytically as elastic bodies. These are often studied through the use of their equivalent discrete systems. (4) The assumption of lumped parameters is not to replace the basic understanding of a problem, but it simplifies the analytical effort and renders a technique for the computer solution.

So far, we have discussed only systems with rectilinear motion. For systems with rotational motions, the elements are (1) the mass moment of inertia of the body J, (2) the torsional spring with spring constant k_t, and (3) the torsional damper with torsional damping coefficient c_t. An angular displacement θ is analogous to a rectilinear displacement x, and an excitation torque $T(t)$ is analogous to an excitation force $F(t)$. The two types of systems are compared as shown in Table 1-1. The comparison is

TABLE 1-1. Comparison of Rectilinear and Rotational Systems

RECTILINEAR	ROTATIONAL
Spring force $= kx$	Spring torque $= k_t\theta$
Damping force $= c\frac{dx}{dt}$	Damping torque $= c_t\frac{d\theta}{dt}$
Inertia force $= m\frac{d^2x}{dt^2}$	Inertia torque $= J\frac{d^2\theta}{dt^2}$

shown in greater detail in Tables 2-2 and 2-3. It is apparent from the comparison that thp concept of rectilinear systems can be extended easily to rotational systems.

1-3 EXAMPLES OF VIBRATORY MOTIONS

To illustrate different types of vibratory motion, let us choose various combinations of the four elements shown in Fig. 1-1 to form-simple dynamic systems.

The spring-mass system of Fig. 1-3(*a*) serves to illustrate the case of undamped free vibration. The mass m is initially at rest at its static equilibrium position. It is acted upon by two equal and opposite forces, namely, the spring force, which is equal to the product of the spring constant k and the static deflection δ_{st} of the spring, and the gravitational force mg due to the weight of the mass m. Now assume that the mass is displaced from equilibrium by an amount x_0 and then released with zero initial velocity. As shown in the free-body sketch, at the time the mass is released, the spring force is equal to $k(x_0 + \delta_{st})$. This is greater than the gravitational force on the mass by the amount kx_0. Upon being released, the mass will move toward the equilibrium position.

Since the spring is initially deformed by x_0 from equilibrium, the corresponding potential energy is stored in the spring. The system is conservative because there is no damper to dissipate the energy. When the mass moves upward and passes through equilibrium, the potential energy of the system is zero. Thus, the potential energy is transformed to become the kinetic energy of the mass. As the mass moves above the equilibrium position, the spring is compressed and thereby gaining potential energy'from the kinetic energy of the mass. When the mass is at its uppermost position, its velocity is zero. All the kinetic energy of the mass has been transformed to become potential energy. Through the exchange of potential and kinetic energies between the spring and the mass, the system oscillates periodically at its natural frequency about its static

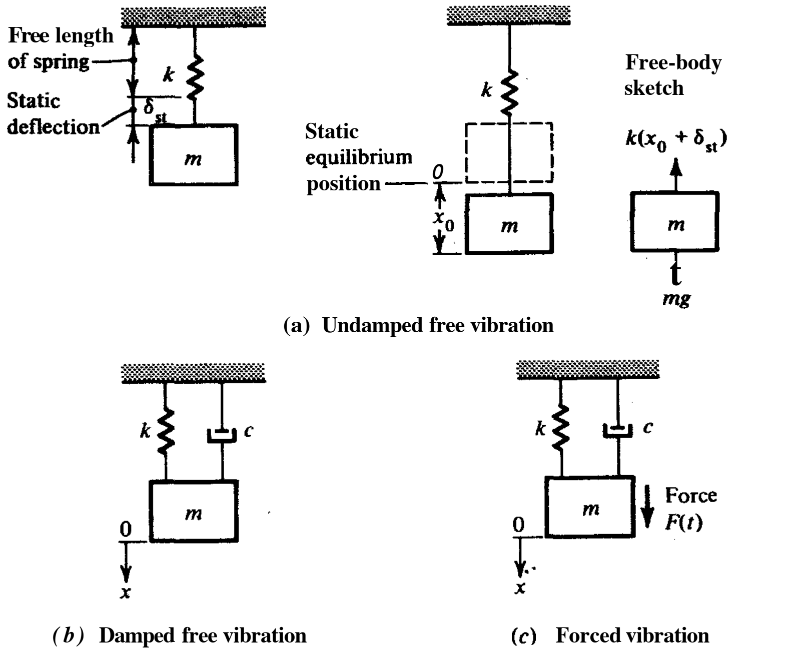


FIG. 1-3. *Simple vibratory systems.*

equilibrium position. Hence natural frequency describes the rate of energy exchange between two types of energy storage elements, namely, the mass and the spring.

It will be shown in Chap. 2 that this periodic motion is sinusoidal or ***simple harmonic.*** Since the system is conservative, the maximum displacement of the mass from equilibrium, or the *amplitude* of vibration, will not diminish from cycle to cycle. It is implicit in this discussion that the natural frequency is a property of the system, depending on the values of m and k. It is independent of the initial conditions or the amplitude of the oscillation.

A mass-spring system with damping is shown in Fig. 1-3(*b*). The mass at rest is under the influence of the spring force and the gravitational force, since the damping force is proportional to velocity. Now, if the mass is displaced by an amount x_0 from its static equilibrium position and then released with zero initial velocity, the spring force will tend to restore the mass to equilibrium as before. In addition to the spring force, however, the mass is also acted upon by the damping force which opposes its motion. The resultant motion depends on the amount of damping in the system. If the damping is light, the system is said to be ***underdamped*** and the motion is oscillatory. The presence of damping will cause (1) the eventual dying out of the oscillation and (2) the system to oscillate more slowly than without damping. In other words, the amplitude decreases with each subsequent cycle of oscillation, and the frequency of vibration with viscous damping is lower than the undamped natural frequency. If the damping is heavy, the motion is nonoscillatory, and the system is said to be ***overdamped.*** The mass, upon being released, will simply tend to return to its static equilibrium position. The system is said to be critically damped if the amount of damping is such that the resultant motion is on the border line between the two cases enumerated. The free vibrations of the systems shown in Figs. 1-3(a) and (b) are illustrated in Fig. 1-4.

Ail physical systems possess damping to a greater or a lesser degree. When there is very 'little damping in a system, such as a steel structure or a simple pendulum, the damping may be negligibly small. Most mechanical systems possess little damping and can be approximated as undamped systems. Damping is often built into a system to obtain the desired performance. For example, vibration-measuring instruments are often built with damping corresponding to 70 percent of the critically damped value.

If an excitation force is applied to the mass of the system as shown in Fig. **1-3(c),** the resultant motion depends on the initial conditions as well as the excitation. In other words, the motion depends on the manner by which the energy is applied to the system. Let us assume that the excitation is sinusoidal for this discussion. Once the system is set into motion, it will tend to vibrate at its natural frequency as well as to follow the **frequency** of the excitation. If the system possesses damping, the part of the motion not sustained by the sinusoidal excitation will eventually die out. This is the transient motion, which is at the natural frequency of the system, that is, the oscillation under free vibrations.

The motion sustained by the sinusoidal excitation is called the steady-state ***vibration*** or the steady-state response. Hence the steady-state response must be at the excitation frequency regardless of the initial

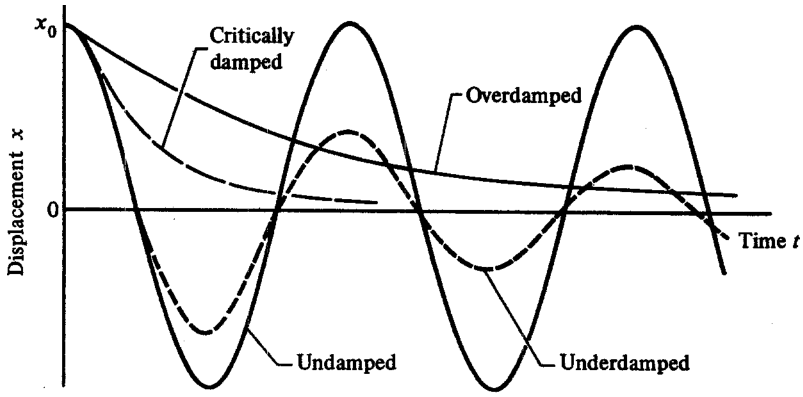


FIG. 1-4. Free vibration of systems shown in Figs. 1-3(a) *and (b). Initial displacement* $= x_o$; *initial velocity* $= 0$.

conditions or the natural frequency of the system. It **will** be shown in Chap. 2 that the steady-state response is described by the particular integral and the transient motion by the complementary function of the differential equation of the system.

Resonance occurs when the excitation frequency is equal to the natural frequency of the system. No energy input is needed to maintain the vibrations of an undamped system at its natural frequency. Thus, any energy input will be used to build up the amplitude of the vibration, and the amplitude at resonance of an undamped system **will** increase without limit. In a system with damping, the energy input is dissipated in the damper. Under steady-state condition, the net energy-input per cycle is equal to the energy dissipation per cycle. Hence the **amplitude** of vibration at resonance for systems with damping is Mite, and it is determined by the amount of damping in the system.

1-4 SIMPLE HARMONIC MOTION

Simple harmonic motion is the simplest form of periodic motion. It will be shown in later chapters that (1) harmonic motion is also the basis for more complex analysis using Fourier technique, and (2) steady-state analysis can be greatly simplified using vectors to represent harmonic motions. We shall discuss simple harmonic motions and the manipulation of vectors in some detail in this section.

A ***simple harmonic motion*** is a ***reciprocating motion.*** It can be represented by the circular functions, sine or cosine. Consider the motion of the point P on the horizontal axis of Fig. 1-5. If the distance OP is

$$OP = x(t) = X \cos \omega t \tag{1-1}$$

where t = time, ω = constant, and X = constant, the motion of P about the origin O is sinusoidal or ***simple harmonic.**** Since the circular function repeats itself in 2π radians, a cycle of motion is completed when $\omega\tau = 2\pi$,

* A sine, a cosine, or their combination can be used to represent a simple harmonic motion. For example, let

$$x(t) = X_1 \sin \omega t + X_2 \cos \omega t = X\left(\frac{X_1}{X} \sin \omega t + \frac{X_2}{X} \cos \omega t\right)$$

$$= X(\sin wt \cos a + w s\, \omega t \sin a) = X \sin(\omega t + a)$$

where $X = \sqrt{X_1^2 + X_2^2}$ and $a = \tan^{-1}(X_2/X_1)$. It is apparent that the motion $x(t)$ is sinusoidal and, therefore, simple harmonic. For simplicity, we shall confine our discussion to a cosine function.

In Eq. (1-1), $x(t)$ indicates that x is a function of time t. Since this is implicit in the equation, we shall omit (t) in all subsequent equations.

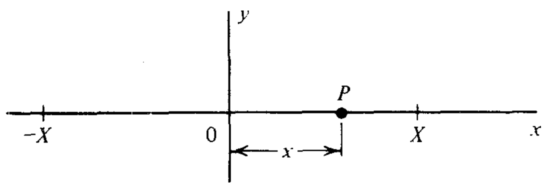


FIG. 1-5. Simple *harmonic motion:* $x(t) = X \cos \omega t$.

that is,

$$\text{Period } \tau = \frac{2\pi}{\omega} \text{ s/cycle} \tag{1-2}$$

$$\text{Frequency } f = \frac{1}{7} = \frac{\omega}{2\pi} \text{ cycle/s, or Hz*} \tag{1-3}$$

ω is called the circular frequency measured in rad/s.

If $x(t)$ represents the displacement of a mass in a vibratory system, the velocity and the acceleration are the first and the second time derivatives of the displacement,? that is,

$$\text{Displacement } x = X \cos \omega t \tag{1-4}$$

$$\text{Velocity } x = -\omega X \sin \omega t = \omega X \cos(\omega t + 90") \tag{1-5}$$

$$\text{Acceleration } \ddot{x} = -\omega^2 X \cos \omega t = \omega^2 X \cos(\omega t + 180°) \tag{1-6}$$

These equations indicate that the velocity and acceleration of a harmonic displacement are also harmonic of the same frequency. Each differentiation changes the amplitude of the motion by a factor of ω and the phase angle of the circular function by 90°. The phase angle of the velocity is 90" leading the displacement and the acceleration is 180° leading the displacement.

Simple harmonic motion can be defined by combining Eqs. (1-4) and (1-6).

$$\ddot{x} = -\omega^2 x \tag{1-7}$$

where ω^2 is a constant. When the acceleration of a particle with rectilinear motion is always proportional to its displacement from a fixed point on the path and is directed towards the fixed point, the particle is said to have simple harmonic motion. It can be shown that the solution of Eq. (1-7) has the form of a sine and a cosine function with circular frequency equal to w.

* In 1965, the Institute of Electrical and Electronics Engineers, Inc. (IEEE) adopted new standards for symbols and abbreviation (IEEE Standard No. 260). The unit :hertz (Hz) replaces cycles/sec (cps) for frequency. Hz is now commonly used in vibration studies.

† The symbols x and $\ddot{x}$ represent the first and second time derivatives of the function $x(t)$, respectively. This notation is used throughout the text unless ambiguity may arise.

The sum of two harmonic functions of the same frequency but with different phase angles is also a harmonic function of the same frequency. For example, the sum of the harmonic motions $x_1 = X_1 \cos \omega t$ and $x_2 = X_2 \cos(\omega t + \alpha)$ is

$$\begin{aligned} x = x_1 + x_2 &= X_1 \cos \omega t + X_2 \cos(\omega t + \alpha) \\ &= X_1 \cos \omega t + X_2(\cos \omega t \cos \alpha - \sin \omega t \sin \alpha) \\ &= (X_1 + X_2 \cos \alpha)\cos \omega t - X_2 \sin \alpha \sin \omega t \\ &= X(\cos \beta \cos \omega t - \sin \beta \sin \omega t) \\ &= X \cos(\omega t + \beta) \end{aligned}$$

where $X = \sqrt{(X_1 + X_2 \cos \alpha)^2 + (X_2 \sin \alpha)^2}$ is the amplitude of the resultant harmonic motion and $\beta = \tan^{-1}(X_2 \sin \alpha)/(X_1 + X_2 \cos \alpha)$ is its phase angle.

The **sum** of two harmonic motions of different frequencies is not harmonic. **A** special case of interest is when the frequencies are slightly different. Let the sum of the motions x_1 and x_2 be

$$\begin{aligned} x = x_1 + x_2 &= X \cos \omega t + X \cos(\omega + \varepsilon)t \\ &= X[\cos \omega t + \cos(\omega + \varepsilon)t] \\ &= 2X \cos\frac{\varepsilon}{2} t \cos\left(\omega + \frac{\varepsilon}{2}\right)t \end{aligned}$$

where $\varepsilon \ll \omega$. The resultant motion $x(t)$ may be considered as a cosine wave with the circular frequency $(\omega + \varepsilon/2)$, which is approximately equal to ω, and with a varying amplitude $[2X \cos(\varepsilon/2)t]$. The resultant motion is illustrated in Fig. 1-6. Every time the amplitude reaches a maximum, there is said to be a beat. The beat frequency f_b, as determined by two

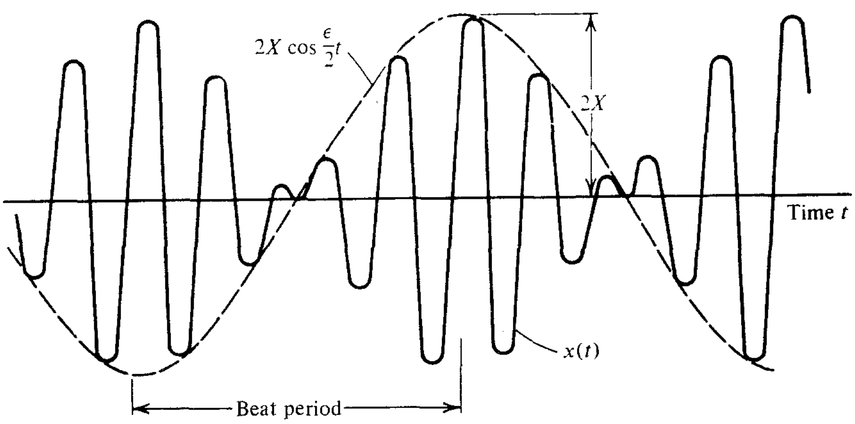


FIG. 1-6. *Graphical representation of beats.*

consecutive maximum amplitudes, is

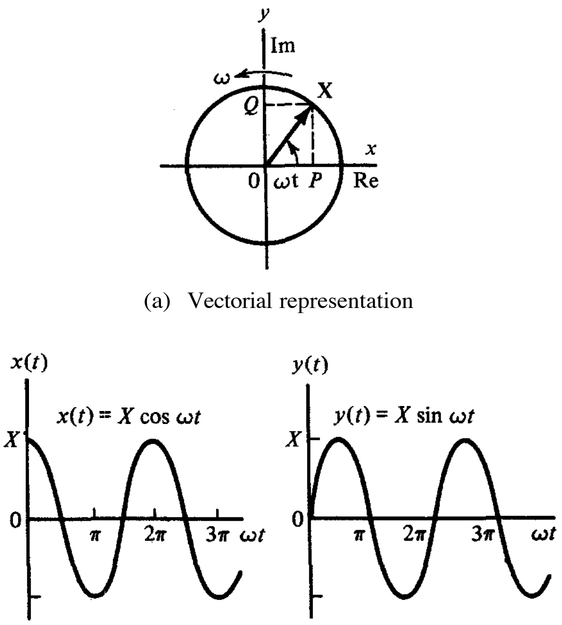
$$f_b = f_2 - f_1 = \frac{\omega + \varepsilon}{2\pi} - \frac{\omega}{2\pi} = \frac{\varepsilon}{2\pi} \tag{1-8}$$

where f_1 and f_2 are the frequencies of the constituting motions. The more general case, for which the amplitudes of x_1 and x_2 are unequal, is left as an exercise.

The phenomenon of beats is common in engineering. Evidently beating can be a useful technique in frequency measurement in which an unknown frequency is compared with a standard frequency.

1-5 VECTORIAL REPRESENTATION OF HARMONIC MOTIONS

It is convenient to represent a harmonic motion by means of a rotating *vector* **X** of constant magnitude* X at a constant angular velocity ω. In Fig. 1-7, the displacement of P from the center O along the x axis



(a) Vectorial representation

(b) Harmonic motions

FIG. 1-7. Harmonic motions represented by rotating vector.

* In complex variables, the length of a vector is called the ***absolute value*** or ***modulus,*** and the phase angle is called the ***argument*** or ***amplitude.*** The length of the vector in this discussion is the amplitude of the **harmonic** motion. To avoid confusion, we shall use ***magnitude*** to denote the length of the vector.

is $OP = x(t) = X$ cos ot. This is the projection of the rotating vector **X** on the diameter along the x axis. Similarly, the projection of **X** on the y axis is $OQ = y(t) = X \sin \omega t$. Naming the x axis as the "real" axis and the y axis as the "imaginary" one, the rotating vector **X** is represented by the equation*

$$\mathbf{X} = X \cos ot + jX \sin \omega t = Xe^{iu'} \qquad \textbf{(1-9)}$$

where X is the length of the vector or its magnitude and $j = \sqrt{-1}$ is called the imaginary unit.

If a harmonic function is given as $x(t) = X \cos \omega t$, it can be expressed as $x(t) = \text{Re}[Xe^{j\omega t}]$, where the symbol Re denotes the real part of the function $Xe^{j\omega t}$. Similarly, the function $y(t) = X$ sin wt can be expressed as y(t)=Im[Xe$^{\cdot u'}$], where the symbol Im denotes the imaginary part of $Xe^{j\omega t}$. It should be remembered that a harmonic motion is a reciprocating motion. Its representation by means of a rotating vector is only a convenience. This enables the exponential function $e^{j\omega t}$ to be used in equations involving harmonic functions. The use of complex functions and complex numbers greatly simplifies the mathematical manipulations of this type of equations. In reality, all physical quantities, whether they are displacement, velocity, acceleration, or force, must be real quantities.

The differentiation of a harmonic function can be carried out in its vectorial form. The differentiation of a vector **X** is

$$\begin{aligned}\frac{d}{dt}\mathbf{X} &= \frac{d}{dt}(Xe^{j\omega t}) = j\omega Xe^{j\omega t} = j\omega\mathbf{X}\\ \frac{d^2}{dt^2}\mathbf{X} &= \frac{d}{dt}(j\omega Xe^{j\omega t}) = (j\omega)^2 Xe^{j\omega t} = -\omega^2\mathbf{X}\end{aligned} \qquad \textbf{(1-10)}$$

* A complex number z is of the form $z = x + jy$, where x is the real part and y the imaginary part of z. Both x and y may be time dependent. For a specific time, x and y are numbers and z can be treated as a complex number. Let X in Fig. 1-7(a) be a complex number. The vector X is

$$\mathbf{X} = x + jy = X(x/X + jy/X) = X(\cos \omega t + j \sin \omega t)$$

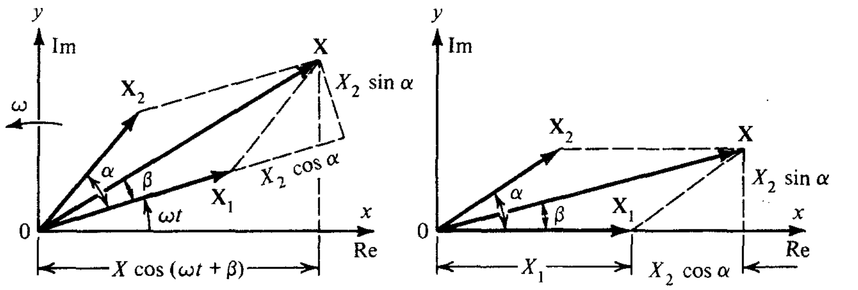
where $X = \sqrt{x^2 + y^2}$ is the magnitude of the vector X. Defining $\theta = \omega t$ and expanding the sine and cosine functions by Maclaurin's series, we obtain

$$\begin{aligned}\mathbf{X} &= X(\cos\theta + j\sin\theta)\\ &= X\left[\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\right]\\ &= X\left(1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \cdots\right)\\ &= X\left[1 + \frac{(j\theta)}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \cdots\right]\\ &= Xe^{j\theta} = Xe^{j\omega t}\end{aligned}$$

The equation $e^{\pm j\theta} = \cos 0 \pm j \sin 0$ is called Euler's formula.

contributed to this field of study, and to the authors listed in the references. We are especially grateful to Dr. James L. Klemm for his suggestions in Chapter 9, and to K. G. Mani for his contribution of the subroutine $PLOTF in Appendix C.

Francis S. Tse
Ivan E. Morse
Rolland T. Hinkle



(a) Vector addition (*b*) Vector addition with phasor notation ($wt = 0$)

FIG. 1-9. Addition of harmonic functions: vectorial method.

with respect to $\mathbf{X}_1$. Since the original motions are given along the real axis, the sum of the harmonic motions is $\text{Re}[\mathbf{X}] = X\cos(\omega t + \beta)$. The addition operation can readily be extended to include the subtraction operation.

Since both $\mathbf{X}_1$ and $\mathbf{X}_2$ are rotating with the same angular **velocity** ω, only the relative phase angle of the vectors is of interest. It is convenient to assign arbitrarily $wt = 0$ as a datum of measurement of phase angles. The vector $\mathbf{X}_1$, $\mathbf{X}_2$, and their sum X are plotted in this manner in Fig. 1-9(*b*). Note that the vector $\mathbf{X}_2$ can be expressed as

$$\mathbf{X}_2 = X_2 e^{j(\omega t + \alpha)} = (X_2 e^{j\alpha}) e^{j\omega t} = (X_2 \cos \alpha + jX_2 \sin \alpha) e^{j\omega t}$$

or

$$\mathbf{X}_2 = \bar{X}_2 e^{j\omega t} \tag{1-12}$$

The quantity $\bar{X}_2 = X_2 e^{j\alpha}$ is a complex number and is called the complex amplitude or phasor of the vector X,. Similarly, $\bar{X} = Xe^{j\beta}$ in Fig. 1-9(*b*) is the phasor of the vector X.

Harmonic functions can be added algebraically by means of vector addition. Using the same functions $x_1 = X_1 \cos wt$ and $x_2 = X_2 \cos(\omega t + \text{a})$, their vector sum is

$$\begin{aligned}\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 &= X_1 e^{j\omega t} + X_2 e^{j(\omega t + \alpha)} = (X_1 + X_2 e^{j\alpha}) e^{j\omega t} \\ &= (X, + X_2 \cos \text{a} + jX_2 \sin \alpha) e^{j\omega t} \\ &= Xe^{j\beta} e^{j\omega t} = Xe^{j(\omega t + \beta)}\end{aligned}$$

where

$$X = \sqrt{(X_1 + X_2 \cos \text{a})^2 + (X_2 \sin \text{a})^2}$$

and

$$\beta = \tan^{-1} \frac{X_2 \sin \text{a}}{X_1 + X_2 \cos \text{a}}$$

Since the given harmonic motions are along the real axis, their sum is

$$x = \text{Re}[\mathbf{X}] = \text{Re}[Xe^{j(\omega t+\beta)}] = X\cos(\omega t+\beta)$$

In representing harmonic motions by rotating vectors, it is often necessary to determine the product of complex numbers. The product can be found by expressing the complex numbers in the exponential form. For example, the product of the complex numbers $\bar{A}$ and $\bar{B}$ is

$$\bar{A}\bar{B} = (a_1 + ja_2)(b_1 + jb_2)$$

$$\bar{A}\bar{B} = (Ae^{j\alpha})(Be^{j\beta}) = ABe^{j(\alpha+\beta)} \tag{1-13}$$

where $A = \sqrt{a_1^2 + a_2^2}$ and $B = \sqrt{b_1^2 + b_2^2}$ are the magnitudes of the numbers and $\alpha = \tan^{-1} a_2/a_1$ and $\beta = \tan^{-1} b_2/b_1$ are their phase angles. Equation (1-13) indicates that

$$\text{Magnitude of } \bar{A}\bar{B} = (\text{magnitude of } \bar{A})(\text{magnitude of } \bar{B}) \tag{1-14}$$

$$\text{Phase of } \bar{A}\bar{B} = (\text{phase of } \bar{A}) + (\text{phase of } \bar{B}) \tag{1-15}$$

Obviously, the multiplication operation can be generalized to include the division operation.

Example 1. Manipulation of Complex Numbers

(a) $\bar{A} = 1 + j\sqrt{3} = \sqrt{1+3}(\cos 60° + j\sin 60°) = 2e^{j\pi/3} = 2\underline{/60°}$
The symbol $2\underline{/60°}$ is a convenient way of writing $2e^{j\pi/3}$. It represents a vector of magnitude of two units and a phase angle of 60° or $\pi/3$ rad counter-clockwise *relative* to the reference x axis.

(b) $\bar{A} = \bar{A}_1 + \bar{A}_2 = (1+j2) + (4+j3) = 5 + j5 = 5\sqrt{2}\underline{/45°}$

(c) $\bar{A} = \bar{A}_1\bar{A}_2 = (1+j\sqrt{3})(4+j3) = (2e^{j\pi/3})(5e^{j0.642})$
$= 10e^{j(\pi/3+0.642)} = 10\underline{/60°+36.8°} = 10\underline{/96.8°}$

(d) $\bar{A} = \dfrac{\bar{A}_1}{\bar{A}_2} = \dfrac{1+j\sqrt{3}}{4+j3} = \dfrac{2\underline{/60°}}{5\underline{/36.8°}} = \dfrac{2}{5}\underline{/60°-36.8°} = 0.4\underline{/23.2°}$

(e) $\bar{A} = 2j = 0 + j2 = 2\left(\cos\dfrac{\pi}{2} + j\sin\dfrac{\pi}{2}\right) = 2e^{j\pi/2} = 2\underline{/90°}$

(f) $\bar{A} = j(1+j\sqrt{3}) = (e^{j\pi/2})(2e^{j\pi/3}) = 2\underline{/90°+60°} = 2\underline{/150°}$

(g) $\bar{A} = \dfrac{1+j\sqrt{3}}{j} = \dfrac{2\underline{/60°}}{1\underline{/90°}} = 2\underline{/60°-90°} = 2\underline{/-30°}$

The last two examples indicate that the multiplication of a vector by j advances the vector counter-clockwise by a phase angle of 90° and a division by j retards it by 90°.

1-6 UNITS

Since there will be a change from the English engineering (customary) to the International System of Units (SI), the two systems of units will co-exist for some years. The student and the practicing engineer will need to know both systems. We shall briefly discuss the English units and then the SI units in some detai; in this section.

Newton's law of motion may be expressed as

$$\text{Force} = (\text{mass})(\text{acceleration}) \tag{1-16}$$

Dimensional homogeneity of the equation is obtained when the force is in pounds lb_f, the mass in slugs, and the acceleration in ft/sec^2. This is the English ft-lb_f-sec system in which the mass has the unit of lb_f-sec^2/ft. A body falling under the influence of gravitation has an acceleration of g ft/sec^2, where $g \simeq 32.2$ is the gravitational acceleration. Hence one pound-mass lb_m exerts one pound-force lb_f under the gravitational pull of the earth. In other words, 1 lb, weighs one pound on a spring scale. If a mass is given in pounds, lb_m or weight, it must be divided by g to obtain dimensional homogeneity in Eq. (1-16).

The in.-lb_f-sec system is generally used in the study of vibrations. The gravitational acceleration is $(32.2)(12) = 386$ in./sec^2. Hence the weight is divided by 386 in order to obtain dimensional homogeneity in Eq. (1-16). We assume that the gravitational acceleration is constant unless otherwise stated. In the derivation of equations, the mass m is assumed to have the proper units.

The International System of Units (SI) is the modernized version of the metric system.*- SI consists of (1) seven well-defined base units, (2) derived units, and (3) supplementary units.

The ***base units*** are regarded as dimensionally independent. Those of interest in this study are the meter, the kilogram, and the second. The ***meter*** m is the unit of length. It is defined in terms of the wave-length of a krypton-85 lamp as "the length equal to 1650 763.73 wave-lengths in vacuum of the radiation corresponding to the transition between the levels $2p_{10}$ and $5d_5$ of the krypton-86 atom." The ***kilogram*** kg is the unit

*** A description of SI and a brief bibliography can be found in the literature. A kit, containing the above and several other publications, is obtainable from the American Society of Engineering Education:**

***The International System of Units (SI)*, Edited by C. H. Page and P. Vigoureux, US National Bureau of Standards, Special Publication 330, Revised 1974.**

***ASME Guide SI-l, ASME Orientation and Guide of SI* (Metric) *Units,* 6th ed., ASME, United Engineering Center, 345 E. 47th St., New York, N.Y. 10017, 1975.**

***Some References on Metric Information,* US National Bureau of Standards, Special Publication 389, Revised 1974.**

ASEE Metric (SI) Resource Kit Project, One Dupont Circle, Suite 400, Washington, DC 20036.

TABLE 1-2(*a*). *Examples of SI Derived Units*

	SI UNIT			
QUANTITY	NAME	SYMBOL	IN TERMS OF SI BASE UNITS	IN TERMS OF OTHER UNITS
Area	square meter	m^2		
Volume	cubic meter	m^3		
Speed, velocity	meter per second	m/s		
Acceleration	meter per second squared	m/s^2		
Density, mass density	kilogram per cubic meter	kg/m^3		
Specific volume	cubic meter per kilogram	m^3/kg		
Frequency	hertz	Hz	s^{-1}	
Force	newton	N	$m \cdot kg \cdot s^{-2}$	
Pressure, stress	pascal	Pa	$m^{-1} \cdot kg \cdot s^{-2}$	N/m^2
Energy, work	joule	J	$m^2 \cdot kg \cdot s^{-2}$	$N \cdot m$
Power	watt	W	$m^2 \cdot kg \cdot s^{-3}$	J/s
Moment of force	meter newton	$N \cdot m$	$m^2 \cdot kg \cdot s^{-2}$	

TABLE 1-2(*b*). *SI Supplementary Units*

	SI UNIT	
QUANTITY	NAME	SYMBOL
Plane angle	radian	rad
Solid angle	steradian	sr

of mass. The standard is a cylinder of platinum-iridium, called the International Standard, kept in a vault at Sévres, France. The *second* s is the unit of time. It is defined in terms of the frequency of atomic resonators. "The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom."*

The derived units are formed from the base units according to the algebraic relations linking the corresponding quantities. Several derived units are given special names and symbols. The *supplementary units* form a third class of SI units. Examples of derived units and the supplementary units are shown in Tables 1-2(*a*) and (*b*), respectively.

*Page and Vigoureux, *op. cit.*, p. 3.

TABLE 1-3. *Prefixes for Multiples and Submultiples of SI Units*

MULTIPLE	PREFIX	SYMBOL	SUBMULTIPLE	PREFIX	SYMBOL
10^{12}	tera	T	10^{-1}	deci	d
10^{9}	giga	G	10^{-2}	centi	c
10^{6}	mega	M	10^{-3}	milli	m
10^{3}	kilo	k		micro	μ
10^{2}	hecto	h	10^{-9}	nano	n
10	deca	dc	10^{-12}	pico	P
			10^{-15}	femto	f
			10^{-18}	atto	a

The common prefixes for multiples and submultiples of SI units are shown in Table 1-3. Examples of conversion from the English to the SI units are given in Table 1-4. Note that a common error in conversion is to become ensnared in too many decimal places. The result of a computation cannot have any more significant numbers than that in the original data.

For uniformity in the use of SI units, the recommeiidations* are: (1) In *numbers,* a period (dot) is used only to separate the integral part of numbers from the decimal part. Numbers are divided into groups of three to facilitate reading. For example, in defining the meter above, we have "...equal to 1 650 763.73 wave-lengths ..." (2) The *type* used for symbols is illustrated in Table 1-2. The lower case roman type is generally used. If the symbol is derived from a proper name, capital roman type is

TABLE 1-4. *Examples of English Units to SI Conversion*[a]

TO CONVERT FROM	TO	MULTIPLY BY
Inch (in.)	meter (m)	2.540 000 E−02
Pound-mass (lb_m)	kilogram (kg)	4.535 924 E−01
Pound-mass/inch^3 (lb_m/in.^{3})	kilogram/meter^3(kg/m^3)	2.767 990 E+04
Slug	kilogram (kg)	1.459 390 E+01
Pound-force (lb_f)	newton (N)	4.448 222 E+00
Pound-force-inch (lb_f-in.)	newton-meter (N . m)	1.129 848 E−01
Pound-force/inch (lb_f/in.)	newton/meter (N/m)	1.751 268 E+02
Pound-force/inch^2 (lb_f/in.^{2})	pascal (Pa)	6.894 757 E+03
Horsepower (550 ft-lb_f/sec)	watt (W)	7.456 999 E+02

[a]The table gives the conversion from the in.-lb_f-sec units to the SI units. The second and radian are commonly used in both systems and no conversions are needed. For example, the damping coefficient c from Table 2-2 has the units of lb_f-sec/in. The value of c is multiplied by 175.1 to obtain the value of c in N · s/m.

*Page and Vigoureux, op. *cit.*, p. 10.

used for the first letter. The symbols are not followed by a period. (3) The ***product of units*** is denoted by a dot, such as N . m shown in Table 1-2. The dot may be omitted if there is no risk of confusion with another unit symbol, such as N m but not mN. (4) The ***division of units*** may be indicated by a solidus (/), a horizontal tine, or a negative power. For example, velocity in Table 1-2 can be expressed as m/s, $\frac{m}{s}$, or m . s^{-1}. The solidus must not be repeated on the same line unless ambiguity is avoided by parentheses. For example, acceleration may be expressed as m/s^2 or m . s^{-2} but not m/s/s. (5) The prefix ***symbols*** illustrated in Table 1-3 are used without spacing between the prefix symbol and the unit symbol, such as in mm. Compound prefixes formed by the use of two or more SI prefixes are not used.

1-7 SUMMARY

Some basic concepts and terminology commonly used in vibration are described in this chapter.

The idealized ***model*** of a simple vibratory system in Fig. 1-1 consists of (1) a rigid mass, (2) a linear spring, (3) a viscous damper, and (4) an excitation. The inertia force is equal to the product of the mass and its acceleration as defined by Newton's law of motion. The spring force is proportional to the spring deformation, that is, the relative displacement between the two ends of the spring. The damping force is proportional to the relative velocity between the two ends of the damper. An excitation may be applied to the mass and/or other parts of the system.

If a system is unforced. the energy stored due to the initial conditions will cause it to vibrate about its static equilibrium position. If damping is zero. the system will oscillate at its natural frequency without diminishing in amplitude. If the system is underdamped. the amplitude of oscillation will diminish with each cycle and the frequency is lower than that without damping.

If a periodic excitation is applied to a system. the vibration consists of (1) a steady-state response and (2) a transient motion. 'The former is being sustained by the excitation and is therefore at the excitation frequency. The latter is due to the initial energy stored in the system and is at its damped natural frequency. Resonance occurs when the system is excited at its natural frequency. The amplitude at resonance is limited only by the damping in the system.

A simple harmonic motion is a reciprocating motion as shown in Fig. 1-5. Alternatively, it can be represented by means of a sinusoidal wave or a rotating vector as shown in Figs. 1-7 and 1-8. These representations are artifices for the convenience of visualization and manipulation only. Using these representations, it can be shown that the velocity leads the displacement by 90° and the acceleration leads the velocity by 90". **A** complex amplitude, shown in Fig. 1-9, is called a phasor. It has a magnitude and a phase angle relative to the reference vector.

A complex number has magnitude and direction. It can be added (subtracted) by adding (subtracting) the real and imaginary parts separately. The product (quotient) of complex numbers is determined by Eqs. (1-14) and (1-15).

$$\text{Magnitude of } \bar{A}\bar{B} = (\text{magnitude of } \bar{A})(\text{magnitude of } \bar{B})$$
$$\text{Phase of } \bar{A}\bar{B} = (\text{phase of I?}) + (\text{phase of B})$$

The in.-lb_f-sec system is generally used in vibration. The gravitational constant is $g \simeq 386$ in./sec^2. There will be a change to the International Systems of Units (SI). The gravitational constant in SI units is $g \simeq$ 9.81 m/s^2. Examples of SI units and the conversion from the English to the SI units are given in Tables 1-2 and 1-4, respectively.

PROBLEMS

1-1 Describe, with the aid of a sketch when necessary, each of the following:

(a) Spring force, damping force, inertia force, excitation.

(b) Kinetic energy, potential energy.

(c) Free vibration, forced vibration, a conservative system.

(d) Steady-state response, transient motion.

(e) Discrete system, continuous system.

(f) Natural frequency, resonance.

(g) Initial conditions, static equilibrium position.

(h) Rectilinear motion, rotational motion.

(i) Periodic motion, frequency, period, beat frequency.

(j) Superposition.

(k) Underdamped system, critically damped system.

(I) Amplitude, phasor, phase angle

1-2 **A** harmonic displacement is $x(t) = 10\sin(30t - \pi/3)$ mm, where t is in seconds and the phase angle in radians. Find (a) the frequency and the period of the motion, (b) the maximum displacement, velocity, and acceleration, and (c) the displacement, velocity, and acceleration at $t = 0$ s. Repeat part (c) for $t = 1.2$ s.

1-3 Repeat Prob. 1-2 if the harmonic velocity is $\dot{x}(t) = 150\cos(17t + \pi/2)$ mm/s.

1-4 An accelerometer indicates that the acceleration of a body is sinusoidal at a frequency of 40 Hz. If the maximum acceleration is 100 m/s^2, find the amplitudes of the displacement and the velocity.

1-5 Repeat Prob. 1-4 if the acceleration lags the excitation by 15°. What is the excitation frequency?

1-6 A harmonic motion is described as $x(t) = X\cos(100t + \psi)$ mm. The initial conditions are $x(0) = 4.0$ mm and $\dot{x}(0) = 1.0$ m/s.

(a) Find the constants X and ψ.

(b) Expressing $x(t)$ in the form

$$x = A\cos\omega t + B\sin\omega t$$

and find the constants A and B.

1-7 Given $x(t) = X\cos(100t + \psi) = A\cos 100t + B\sin 100t$, find A, B, X, and ψ for each set of the following conditions:

(a) $x(0.1) = -8.796$ mm and $x(0.2) = 10.762$ mm

(b) $x(0.1) = -8.796$ mm and $\dot{x}(0.1) = -621.5$ mm/s

(c) $x(0.1) = -8.796$ mm and $\ddot{x}(0.2) = -10.76 \times 10^4$ mm/s^2

(d) $x(0) = 4.0$ mm and $\ddot{x}(0.2) = -10.76 \times 10^4$ mm/s^2

1-8 A table has a vertical sinusoidal motion with constant frequency. What is the largest amplitude that the table can have, if an object on the table is to remain in contact?

1-9 Find the algebraic sum of the harmonic motions x_1 and x_2.

$$x = x_1 + x_2 = 2\sin(\omega t + \pi/3) + 3\sin(\omega t + 2\pi/3)$$
$$= X\sin(\omega t + \alpha)$$

Find X and α. Check the addition graphically.

1-10 The motion of a particle is described as $x = 4\sin(\omega t + \pi/6)$. If the motion has two components, one of which is $x_1 = 2\sin(\omega t - \pi/3)$, determine the other harmonic component.

1-11 In a sketch of x versus t for $0 \le t \le 0.4$ s, plot the motions described by each of the equations: $x_1 = 5\sin 10\pi t$; $x_2 = 4\sin(10\pi t + \pi/4)$; $x_3 = 3\sin(10\pi t - \pi/4)$.

1-12 A periodic motion is described by the equation

$$x = 5\sin 2\pi t + 3\sin 4\pi t$$

In a plot of x versus t, sketch the motion for $0 \le t \le 1.5$ s.

1-13 Repeat Prob. 1-12 if

(a) $x = 5\sin(2\pi t + 30°) + 3\sin(4\pi t + 60°)$

(b) $x = 5\sin(2\pi t + 90°) + 3\sin(4\pi t + 180°)$

1-14 Is the motion $x(t) = \cos 10t + 3\cos(10+\pi)t$ periodic?

1-15 Find the period of the functions

(a) $x = 3\sin 3t + 5\sin 4t$

(b) $x = 7\cos^Z 3t$

1-16 Determine the sum of the harmonic motions $x_1 = X_1 \cos$ wt and $x_2 = X_2\cos(\omega+\varepsilon)t$, where $\varepsilon \ll$ w. If beating should occur, find the amplitude and the beat frequency.

1-17 Sketch the motion described by each of the following equations:

(a) $x = 5e^{-2t}\sin(10\pi t + \pi/4)$

(b) $x = 5e^{-2t}\sin(10\pi t + \pi/4) + 7\sin 4\pi t$

for $0 \le t \le 1.0$ s.

1-18 Express the following complex numbers in the exponential form $Ae^{j\theta}$

(a) $1 + j\sqrt{3}$	**(e)** $3/(\sqrt{3} - j)^2$
(b) -2	**(f)** $(\sqrt{3} + j)(3 + 4j)$
(c) $3/(\sqrt{3} - j)$	**(g)** $(\sqrt{3} - j)/(3 - 4j)$
(d) $5j$	**(h)** $[(2j)^2 + 3j + 8]$

1-19 The motion of a particle vibrating in a plane has two perpendicular harmonic components: $x_1 = 2\sin(\omega t + \pi/6)$ and $x_2 = 3\sin\omega t$. Determine the motion of the particle graphically.

1-20 Repeat Prob. 1-19 using $x_1 = 2\sin(2\omega t + \pi/6)$ and $x_2 = 3\sin$ wt.

2

Systems with One Degree of Freedom—Theory

2-1 INTRODUCTION

The one-degree-of-freedom system is the keystone for more advanced studies in vibrations. The system is represented by means of a generalized model shown in Fig. 1-1. The common techniques for the analysis are discussed in this chapter.

Examples of one-degree-of-freedom systems are shown in Fig. 2-1. Though such systems differ in appearance, they all can be represented by the same generalized model in Fig. 1-1. The model serves (1) to unify a class of problems commonly encountered, and (2) to bring into focus the concepts of vibration. The applications to different types of problem will be discussed in the next chapter.

Four mathematical techniques are examined. These are (1) the energy method, (2) Newton's law of motion, (3) the frequency response method, and (4) the superposition theorem. Our emphasis is on concepts rather than on mathematical manipulations.

Since vibration is an energy exchange phenomenon, the simple energy method is first presented. In applying Newton's second law, the system is described by a second-order differential equation of motion. If, the excitation is an analytical expression, the equation can be solved readily by the "classical" method. If the excitation is an arbitrary function, the motion can be found using the superposition theorem. The frequency response method assumes that the excitation is sinusoidal and examines the system behavior over a frequency range of interest.

Note that a system will vibrate in its own way regardless of the method of analysis. The purpose of different techniques is to find the most convenient method to characterize the system and to describe its behavior. We treat Newton's second law and the superposition theorem as

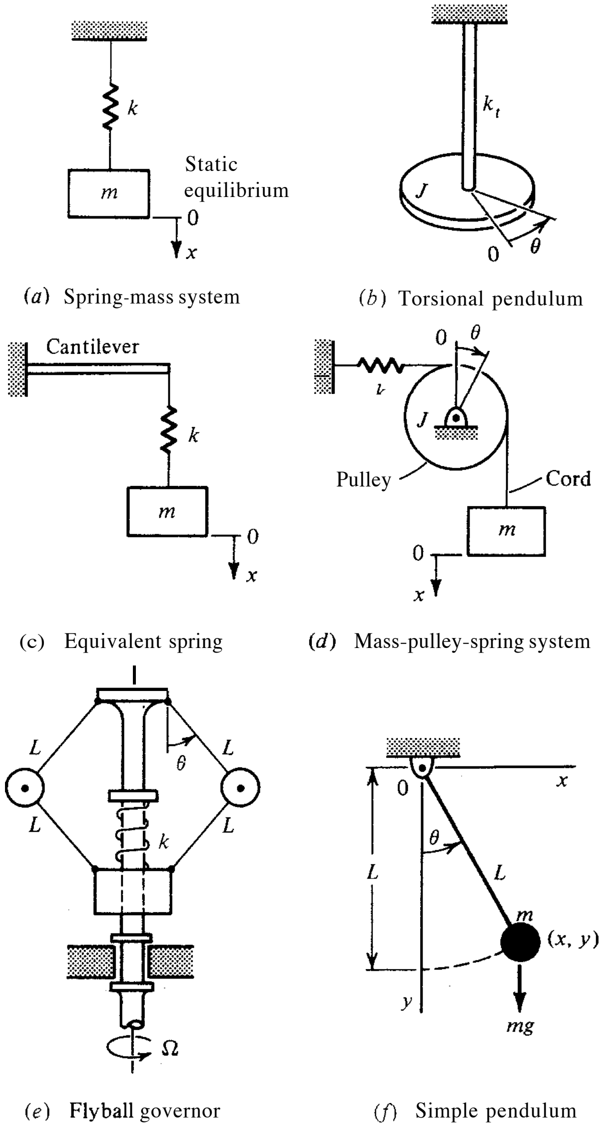


FIG. 2-1. *Examples of systems with one degree of freedom.*

time domain analysis, since the motion of the mass is a time function, such as the solution of a differential equation with time as the independent variable. The frequency response method assumes that both the excitation and the system response are sinusoidal and of the same frequency. Hence it is a ***frequency domain analysis.*** Note that time response is intuitive but it is more convenient to describe a system in the frequency domain.

It should be remarked that there must be a correlation between the time and the frequency domain analyses, since they are different methods to consider the same problem. In fact, superposition, which is treated as a time domain technique, is the basis for the study of linear systems. The convolution integral derived from superposition can be applied in the time or the frequency domain. We are presenting only one aspect of this very important theorem and shall not discuss methods of correlation. The mathematical correlation of the time and frequency analyses is not new. Its implementation, however, was not practical until the advent of computers, instrumentation, and testing techniques in recent years.

2-2 DEGREES OF FREEDOM

The number of ***degrees of freedom*** of a vibratory system is the number of ***independent spatial coordinates*** necessary to define its configuration. A configuration is defined as the geometric location of all the masses of the system. If the inter-relationship of the masses is such that only one spatial coordinate is required to define the configuration, the system is said to possess ***one degree of freedom.***

A rigid body in space requires six coordinates for its complete identification, namely, three coordinates to define the rectilinear positions and three to define the angular rotations. Ordinarily, however, the masses in a system are constrained to move only in a certain manner. Thus, the ***constraints*** limit the degrees of freedom to a much smaller number.

Alternatively, the number of degrees of freedom of a system can be defined as the number of spatial coordinates required to specify its configuration minus the number of ***equations of constraint.**** We shall illustrate these definitions with a number of examples.

The one-degree-of-freedom systems shown in Fig. 2-1 are briefly discussed as follows:

1. The spring-mass system in Fig. 2-1(a) has a mass m suspended from a coil spring with a spring constant k. If m is constrained to move only in the vertical direction about its static equilibrium position 0, only one spatial coordinate $x(t)$ is required to define its configuration. Hence it is said to possess one degree of freedom.
2. The torsional pendulum in Fig. 2-1(b) consists of a heavy disk J and a shaft of negligible mass with a torsional spring constant k_t. If the system is constrained to oscillate about the longitudinal axis of the shaft, the configuration of the system can be specified by a single coordinate $\theta(t)$.

*Such a system is called a holonomic system; it is the only type of system considered in this text. For a discussion on holonomic and nonholonomic systems, see, for example, H. Goldstein, *Classical Mechanics*, Addison-Wesley Publishing Company, Inc., Reading, Mass, 1957, pp. 11–14.

3. The mass-spring-cantilever system in Fig. 2-1(*c*) has one degree of freedom if the cantilever is of negligible mass and the mass ***m*** is constrained to move vertically. By neglecting the inertial effect of the cantilever and considering only its elasticity, the cantilever becomes a spring element. Hence a simple spring-mass system is obtained from the given mass ***m*** and an equivalent spring, constructed from the combination of the spring k and the cantilever.

4. The mass-pulley-spring system in Fig. 2-1(*d*) has one degree of freedom if it is assumed that there is no slippage between the cord and the pulley *J* and the cord is inextensible. Although the system posses-ses two mass elements m and *J*, the linear displacement $x(t)$ of m and the angular rotation $\theta(t)$ of *J* are not independent. Thus, either $x(t)$ or $\theta(t)$ can be used to specify the configuration of the system.

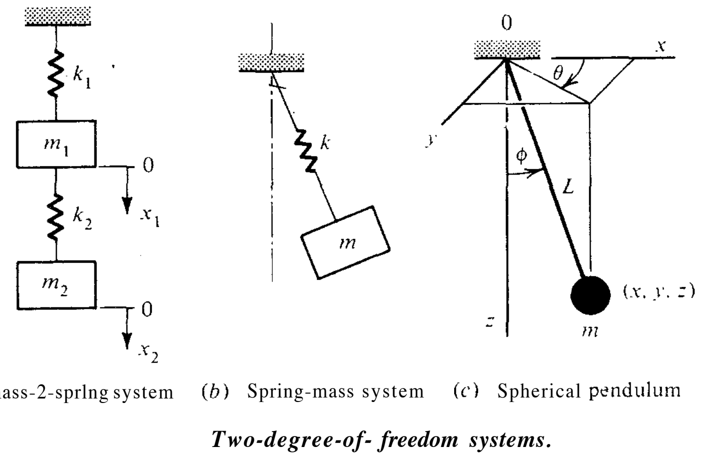
5. **A** simple spring-loaded flyball governor rotating with constant angular velocity Ω is shown in Fig. 2-1(*e*). If a disturbance is applied to the governor, its vibratory motion can be expressed in terms of the angular coordinate $\theta(t)$.

6. The simple pendulum in Fig. 2-1(*f*) is constrained to move in the ***xy*** plane. Its configuration can be defined either by the rectangular Cartesian coordinates $x(t)$ and $y(t)$ or by the angular rotation $\theta(t)$. The (x,y) coordinates, however, are not independent. They are related by the ***equation of constraint***

$$x^2 + y^2 = L^2 \tag{2-1}$$

where the length L of the pendulum is assumed constant. Thus, if $x(t)$ is chosen arbitrarily, $y(t)$ is determined from Eq. (2-1).

Several systems with ***two degrees of freedom*** are shown in Fig. 2-2.



(a) 2-mass-2-sprlng system (*b*) Spring-mass system (*c*) Spherical pendulum

Two-degree-of- freedom systems.

1. The two-spring-two-mass system of Fig. 2-2(*a*) possesses two degrees of freedom if the masses are constrained to move only in the vertical direction. The two spatial coordinates defining the configuration are $x_1(t)$ and $x_2(t)$.

2. The spring-mass system shown in Fig. 2-2(*b*) was described previously as a one-degree-of-freedom system. If the mass m is allowed to oscillate along the axis of the spring as well as to swing from side to side, the system possesses two degrees of freedom.

3. The pendulum in space in Fig. 2-2(*c*) can be described by the $\theta(t)$ and $\phi(t)$ coordinates as well as by the $x(t)$, $y(t)$, and $z(t)$ coordinates. The latter are related by the equation of constraint $x^2 + y^2 + z^2 = L^2$. Thus, this pendulum has only two degrees of freedom.

2-3 EQUATION OF MOTION—ENERGY METHOD

The equation of motion of a conservative system can be established from energy considerations. If a conservative system in Fig. **2-3** is set into motion, its total mechanical energy is the sum of the kinetic energy and the potential energy. The kinetic energy T is due to the velocity of the mass, and the potential energy U is due to the strain energy of the spring by virtue of its deformation. Since the system is conservative, the total mechanical energy is constant and its time derivative must be zero. This can be expressed as

$$T + U = (\text{total mechanical energy}) = \text{constant} \tag{2-2}$$

$$\frac{d}{dt}(T + U) = 0 \tag{2-3}$$

To derive the equation of motion for the spring-mass system of Fig. 2-3, assume that the displacement $x(t)$ of the mass m is measured from its

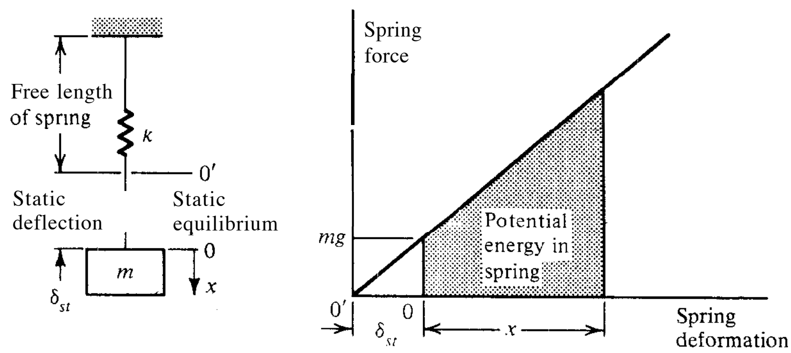


FIG. 2-3. *Potential energy in spring.*

static equilibrium position. Let $x(t)$ be positive in the downward direction. Since the spring element is of negligible mass, the kinetic energy T of the system is

$$T = \tfrac{1}{2}m\dot{x}^2 \tag{2-4}$$

The corresponding potential energy of the entire system is the algebraic sum of (1) the strain energy of the spring and (2) that due to the change in elevation of the mass. The *net* potential energy of the system about the static equilibrium is

$$\begin{aligned} U &= \int (\text{total spring force})\, dx - mgx \\ &= \int_0^x (mg + kx)\, dx - mgx \\ &= \tfrac{1}{2}kx^2 \end{aligned} \tag{2-5}$$

Substituting Eqs. (2-4) and (2-5) into Eq. (2-3) gives

$$\frac{d}{dt}\left(\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2\right) = (m\ddot{x} + kx)\dot{x} = 0$$

Since the velocity $\dot{x}(t)$ in the equation cannot be zero for all values of time, clearly

$$m\ddot{x} + kx = 0 \tag{2-6}$$

$$\ddot{x} + \omega_n^2 x = 0 \tag{2-7}$$

where $\omega_n^2 = k/m$. The equation of motion of the system can be expressed as shown in Eq. (2-6) or (2-7).

It can be shown that the solution of Eq. (2-7) is of the form

$$x = A_1 \cos \omega_n t + A_2 \sin \omega_n t \tag{2-8}$$

where A_1 and A_2 are arbitrary constants to be evaluated by the *initial conditions* $x(0)$ and $\dot{x}(0)$.* It is apparent that ω_n in Eq. (2-7) is the circular frequency of the harmonic motion $x(t)$. Since the components of the solution are harmonic of the same frequency, their sum is also harmonic and can be written as

$$x = A \sin(\omega_n t + \psi) \tag{2-9}$$

where $A = \sqrt{A_1^2 + A_2^2}$ is the amplitude of the motion and $\psi = \tan^{-1} A_1/A_2$ is the phase angle.

Equation (2-9) indicates that once this system is set into motion it will vibrate with simple harmonic motion, and the amplitude A of the motion will not diminish with time. The system oscillates because it possesses two

*The arbitrary constants A_1 and A_2 can be evaluated by conditions specified other than at $t = 0$. It is customary and convenient, however, to use initial conditions.

types of energy storage elements, namely, the mass and the spring. The rate of energy interchange between these elements is the ***natural frequency*** f_n of the system.*

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\ \text{Hz} \qquad \textbf{(2-10)}$$

Note that the natural frequency is a property of the system. It is a function of the values of m and k and is independent of the amplitude of oscillation or the manner by which the system is set into motion. Evidently, only the amplitude **A** and the phase angle ψ are dependent on the initial conditions.

Example 1

Determine the equation of motion of the simple pendulum shown in Fig. 2-1(f).

Solution:

Assume *(1)* the size of the bob is small as compared with the length L of the pendulum and *(2)* the rod connecting the bob to the hinge point 0 is of negligible mass. The mass moment of inertia of the bob of mass m about 0 is

$$J_0 = (J_{cg} + mL^2) \simeq mL^2$$

where J_{cg} is the mass moment of inertia of m about its mass center. If the bob is sufficiently small in size, then $J_{cg} \ll mL^2$.

The angular displacement $\theta(t)$ is measured from the static equilibrium position of the pendulum. The kinetic energy of the system is $T = \frac{1}{2}J_0\dot{\theta}^2 \simeq \frac{1}{2}mL^2\dot{\theta}^2$. The corresponding potential energy is $U = mgL(1-\cos\theta)$, where $L(1-\cos\theta)$ is the change in elevation of the pendulum bob. Substituting these energy quantities in Eq. *(2-3)* gives

$$mL^2\ddot{\theta} + mgL\sin\theta = 0 \qquad \textit{(2-11)}$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \qquad \textbf{(2-12)}$$

The equation of motion of the simple pendulum is as shown in Eq. (2-11) or *(2-12)*. If it is further assumed that the amplitude of oscillation is small, then $\sin\theta \simeq \theta$ and Eq. *(2-12)* becomes

$$\ddot{\theta} + \frac{g}{L}\theta \simeq 0 \qquad \textbf{(2-13)}$$

This is of the same form as Eq. *(2-7)* and the solution follows. The frequency of oscillation of a simple pendulum is $\omega_n = \sqrt{g/L}$.

*It is convenient to call ω_n the natural frequency instead of the natural circular frequency. In the subsequent sections of the text, natural frequency will refer to f_n or ω_n unless ambiguity arises. Similarly, frequency may refer to f or ω.

Note that, if small oscillations are not assumed, Eq. (2-11) is a nonlinear differential equation and elliptical integrals are used for the problem solution. The dependent variable $\theta(t)$ and the independent variable t are related by*

$$t=\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{\dot{\theta}_0^2+\dfrac{2mgL}{J_0}(\cos\theta-\cos\theta_0)}} \tag{2-14}$$

where θ_0 and $\dot{\theta}_0$ are the initial conditions at $t=0$. It is conceivable that, if the pendulum is given a sufficient large initial velocity, the pendulum may continue to rotate about the hinge point. Thus, $\theta(t)$ will increase with time and the motion will not be periodic.

Small oscillations will be assumed throughout this text unless otherwise stated. This assumption greatly simplifies the effort necessary to obtain the solution. Furthermore, the answers will be relevant for most problems, such as in predicting the onset of resonance in a vibratory system.

Example 2

Figure 2-4 shows a cylinder of mass *m* and radius R, rolling without slippage on a curved surface of radius R. Derive the equation of motion of the system by the energy method.

Solution:

The kinetic energy of the cylinder is due to its translational and rotational motions. The translational velocity of the mass center of the cylinder is $(R-R_1)\dot{\theta}$. The angular velocity of the cylinder is $(\dot{\theta}_1-\dot{\theta})$. Since the cylinder

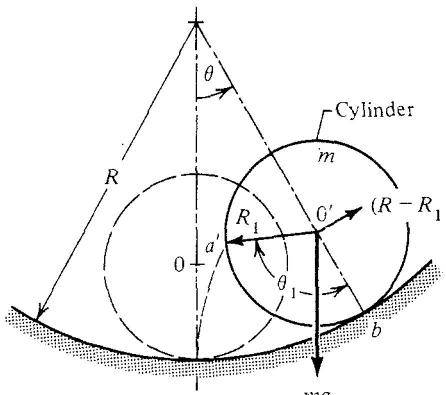


FIG. 2-4. Cylinder on curved surface.

*T. von Karman and M. A. Biot, *Mathematical Methods in Engineering,* McGraw-Hill Book Co., New York, 1940, pp. 115–119.

rolls without slippage, the arc $\widehat{ab} = \text{arc}\,\widehat{a'b}$ and $R\theta = R_1\theta_1$. Hence the angular velocity can be written as $(R/R_1 - 1)\dot{\theta}$. The total kinetic energy T of the cylinder is

$$T = \tfrac{1}{2}m[(R-R_1)\dot{\theta}]^2 + \tfrac{1}{2}J_0[(R/R_1 - 1)\dot{\theta}]^2$$

where $J_0 = \frac{1}{2}mR_1^2$ is the moment of inertia of the cylinder about its longitudinal axis. The potential energy U is due to the change in elevation of the mass center of the cylinder with respect to its static equilibrium position, that is,

$$U = mg(R - R_1)(1 - \cos 8)$$

Substituting the T and U expressions into Eq. (2-3) gives

$$\left[\frac{3}{2}m(R-R_1)^2\ddot{\theta} + mg(R-R_1)\sin\theta\right]\dot{\theta} = 0$$

$$\ddot{\theta} + \frac{2g}{3(R-R_1)}\theta \simeq 0$$

where $\theta \simeq \sin\theta$ for small oscillations. Comparing this with Eq. (2-7), the natural frequency ω_n of the system is equal to $\sqrt{2g/3(R-R_1)}$.

The natural frequency of a conservative system can be deduced by Rayleigh's method. Natural frequency is the rate of energy interchange between the kinetic and the potential energies of a system during its cyclic motion. As the mass passes through the static equilibrium position, the potential energy is zero. Hence the kinetic energy is maximum and is equal to the total mechanical energy of the system. When the mass is at a position of maximum displacement, it is on the verge of changing direction and its velocity is zero. Correspondingly, its kinetic energy is zero. Thus, the potential energy is maximum and is equal to the total mechanical energy of the system. As indicated in Eq. (2-9), the motion is harmonic when the system is vibrating at its natural frequency. The maximum displacement, or amplitude, is A and the maximum velocity is $\omega_n A$. Equating the maximum kinetic and potential energies, we have

$$T_{max} = U_{max} = \text{total energy of the system} \qquad (2\text{-}15)$$

or

$$\tfrac{1}{2}m(\omega_n A)^2 = \tfrac{1}{2}kA^2$$

$$\omega_n = \sqrt{k/m}$$

Example 3. Equivalent mass of spring: Rayleigh method

The mass of the spring shown in Fig. 2-5 is not negligible. Determine ' natural frequency of the system by Rayleigh's method.

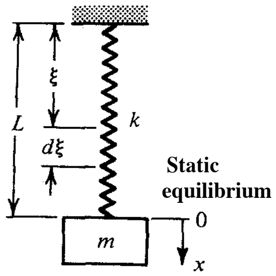


FIG. 2-5. Equivalent *mass* of spring: Rayleigh *method.*

Solution:

Let L be the length of the spring k when the system is at its static equilibrium position. Assume that, when the end of the spring has a displacement $x(t)$, an intermediate point ξ of the spring has a displacement equal to $\frac{5}{L}x(t)$. Thus, $x(t)$ defines the configuration and the system has only one degree of freedom.

The kinetic energy of the system is due to the rigid mass m and the mass of the spring k. The kinetic energy of an element of the spring of length $d\xi$ is $\frac{1}{2}(\rho d\xi)\left(\frac{\xi}{L}\dot{x}\right)^2$, where p = mass/length of the spring. Let $x = A \sin \omega_n t$. Hence the maximum kinetic energy of the system is

$$T_{max} = \tfrac{1}{2}m\dot{x}^2_{max} + \int_0^L \tfrac{1}{2}\rho\left(\frac{\xi}{L}\dot{x}_{max}\right)^2 d\xi$$

$$T_{max} = \tfrac{1}{2}\left(m + \frac{\rho L}{3}\right)\dot{x}^2_{max}$$

$$= \tfrac{1}{2}(m + \rho L/3)(\omega_n A)^2$$

From Eq. (2-5), the maximum potential energy of the system is

$$U_{max} = \tfrac{1}{2}kx^2_{max} = \tfrac{1}{2}kA^2$$

The natural frequency is obtained by equating the maximum kinetic and potential energies, that is,

$$\tfrac{1}{2}(m + \rho L/3)(\omega_n A)^2 = \tfrac{1}{2}kA^2$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m + \rho L/3}}$$

This equation shows that the inertial effect of the spring can be accounted for by adding one-third of the mass of the spring to the rigid mass m. The natural frequency can then be calculated as if the system were to consist of a massless spring and an equivalent rigid mass of $(m + \rho L/3)$.

The approximate method above indicates that the natural frequency is independent of the mass ratio $\rho L/m$, that is, the mass of the spring to that of the rigid mass. For a heavy spring with a light mass, a larger fraction of the spring-mass would have to be used for the frequency calculation. The error, however, is less than one percent, as compared with the exact value, when the spring-mass is equal to the rigid mass?

2-4 EQUATION OF MOTION—NEWTON'S LAW OF MOTION

Newton's law of motion is used to establish the differential equation of mqtion of one-degree-of-freedom systems in this section. The emphasis is on concepts of vibration rather than the technique for solving the equation.

The generalized model representing this class of problems is shown in Fig. 2-6. The displacement $x(t)$ of the mass m is measured from the static equilibrium position. Displacement is positive in the downward direction, and so are the velocity $\dot{x}(t)$ and the acceleration $\ddot{x}(t)$. A positive force on the mass m will produce a positive acceleration of the mass and vice versa. Referring to the free-body sketch, the forces acting on the mass m are (1) the gravitational force mg which is constant, (2) the spring force $k(x+\delta_{st})$ which always opposes the displacement, (3) the damping force $c\dot{x}$ which always opposses the velocity, and (4) the excitation force which is assumed to equal to $F\sin \omega t$.

Newton's law of motion (second law) states that the rate of change of momentum is proportional to the impressed force and takes place in the direction of the straight line in which the force acts. If the mass is constant, the rate of change of momentum is equal to the mass times its acceleration. From the free-body sketch in Fig. 2-6, the equation of motion of the system is

$$m\ddot{x} = \sum(\text{forces})_x \tag{2-16}$$

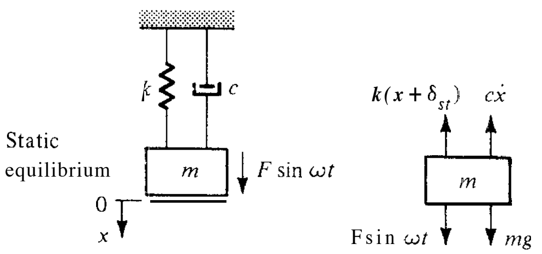


FIG. 2-6. *Model of systems with one degree of freedom.*

*S. Timoshenko and D. Young, *Vibration Problems in Engineering,* 3d ed., D. Van Nostrand Co., Inc., New York, 1954, pp. 306–314.

or

$$m\frac{d^2}{dt^2}(x+\delta_{st}) = -k(x+\delta_{st}) - c\frac{d}{dt}(x+\delta_{st}) + mg + F\sin\omega t \tag{2-17}$$

$$m\ddot{x} + c\dot{x} + kx = F\sin\omega t \tag{2-18}$$

Note that the gravitational force mg is equal to the static spring force $k\delta_{st}$. This is obvious in a simple problem. The implications are (1) that the static forces must cancel in a vibratory system, and (2) that only the dynamic forces need be considered. This concept may be helpful for more complex problems.

The equation above can be derived whether the general position of the mass is considered above or below the static equilibrium position or whether the mass is moving upward or downward. Thus, the equation is true for all time and for all positions of the mass. The verification of this statement is left as an exercise.

Using *d'Alembert's* principle, Eq. (2-16) can be expressed as

$$\sum(\text{forces})_x - m\ddot{x} = 0 \tag{2-19}$$

The quantity $-m\ddot{x}$ is called the inertia force. In other words, introducing the appropriate inertia force, we can say that the impressed force on the mass is in equilibrium with the inertia force. Thus, the dynamic problem is reduced to an "equivalent" problem of statics.

2-5 GENERAL SOLUTION

The equation of motion for the model in Fig. 2-6 is a second-order linear ordinary differential equation with constant coefficients, Eq. (2-18). The general solution $x(t)$ is the sum of the complementary function $x_c(t)$ and the particular integral $x_p(t)$ as shown in Eq. (D-13), App. D.

$$x = x_c + x_p \tag{2-20}$$

Let us consider the two parts of the solution separately before discussing the general solution.

Complementary Function

The complementary function satisfies the corresponding homogeneous equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2-21}$$

The solution is of the form

$$x_c = Be^{st} \tag{2-22}$$

where B and s are constants. Substituting Eq. (2-22) into Eq. (2-21) gives

$$(ms^2+cs+k)Be^{st}=0 \tag{2-23}$$

Since the quantity Be^{st} cannot be zero for all values of t, we deduce that

$$ms^2+cs+k=0 \tag{2-24}$$

This is called the auxiliary or the *characteristic equation* of the system. The roots of the characteristic equation are

$$s_{1,2}=\frac{1}{2m}(-c\pm\sqrt{c^2-4mk}) \tag{2-25}$$

Since there are two roots, the complementary function is

$$x_c=B_1e^{s_1t}+B_2e^{s_2t} \tag{2-26}$$

where B_1 and B_2 are arbitrary constants to be evaluated by the initial conditions.

Let us rewrite the equations above in a more convenient form by defining

$$\frac{k}{m}=\omega_n^2 \qquad \text{and} \qquad \frac{c}{m}=2\zeta\omega_n \qquad \text{or} \qquad \zeta=\frac{c}{2\sqrt{km}} \tag{2-27}$$

where ω_n is the *natural* **circular** *frequency* of the system and ζ is called the *damping* **factor.** Since m, c, and k are positive, ζ is a positive number. Using the definitions of ζ and ω_n, Eqs. (2-21), (2-24), and (2-25) become

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=0 \tag{2-28}$$

$$s^2+2\zeta\omega_ns+\omega_n^2=0 \tag{2-29}$$

$$s_{1,2}=-\zeta\omega_n\pm\sqrt{\zeta^2-1}\,\omega_n \tag{2-30}$$

When $\zeta>1$, Eq. (2-30) shows that the roots are real, distinct, and negative, since $\sqrt{\zeta^2-1}<\zeta$. Thus, no oscillation can be expected from the complementary function in Eq. (2-26) regardless of the initial conditions. Since both the roots are negative, the motion diminishes with increasing time and is aperiodic.

When $\zeta=1$, Eq. (2-30) shows that both the roots are equal to $-\omega_n$. Thus, the complementary function is of the form

$$x_c=(B_3+B_4t)e^{-\omega_nt}$$

where B_3 and B_4 are constants. The motion is again aperiodic. Since the $\lim_{t\to\infty}e^{-\omega_nt}=\lim_{t\to\infty}te^{-\omega_nt}=0$, the motion will eventually diminish to zero.

When $\zeta<1$, the roots are complex conjugates.

$$s_{1,2}=-\zeta\omega_n\pm j\sqrt{1-\zeta^2}\,\omega_n \tag{2-31}$$

where $j=\sqrt{-1}$. Defining

$$\omega_d=\sqrt{1-\zeta^2}\,\omega_n \tag{2-32}$$

and using Euler's formula $e^{\pm j\theta}=\cos\theta\pm j\sin\theta$, the complementary function x_c in Eq. (2-26) becomes

$$x_c=e^{-\zeta\omega_n t}(B_1e^{j\omega_d t}+B_2e^{-j\omega_d t})$$

or

$$x_c=e^{-\zeta\omega_n t}[(B_1+B_2)\cos\omega_d t+j(B_1-B_2)\sin\omega_d t] \tag{2-33}$$

Since the displacement $x_c(t)$ is a real physical quantity, the coefficients, (B_1+B_2) and $j(B_1-B_2)$, in Eq. (2-33) must also be real. This requires that B_1 and B_2 becomplex conjugates. Hence Eq. (2-33) can be rewritten as .

$$x_c=e^{-\zeta\omega_n t}(A_1\cos\omega_d t+A_2\sin\omega_d t) \tag{2-34}$$

or

$$x_c=Ae^{-\zeta\omega_n t}\sin(\omega_d t+\psi) \tag{2-35}$$

where A_1 and A_2 are real constants to be evaluated by the initial conditions. The harmonic functions in Eq. (2-%)..are combined to give Eq. (2-35), where $A=\sqrt{A_1^2+A_2^2}$ and $\psi=\tan^{-1}(A_1/A_2)$. The motion described by Eq. (2-35) consists of a harmonic motion of frequency ω_d and an amplitude $Ae^{-\zeta\omega_n t}$, which decreases exponentially with time.

For the three cases enumerated, the type of motion described by $x_c(t)$ depends on the value of ζ. The system is said to be overdamped when $\zeta>1$, critically damped when $\zeta=1$, and underdamped when $\zeta<1$. This was explained intuitively in Chap. 1. Note that (1) $x_c(t)$ is vibratory only if the system is underdamped; (2) the frequency of oscillation ω_d is lower than the natural frequency ω_n of the system; and (3) in all cases, $x_c(t)$ will eventually die out, regardless of the initial conditions or the excitation. Hence the complementary function gives the transient motion of the system. As a limiting case, if the system has no damping, the amplitude of $x_c(t)$ will not diminish with time. Furthermore, Eq. (2-35) shows that the frequency ω_d and the rate of the exponential decay in amplitude are independent of the arbitrary constants of the equation. In other words, they are properties of the system, independent of the initial conditions or the manner by which the system is set into motion.

The critical damping *coefficient* c_c is the amount of damping necessary for a system to be critically damped, that is, $\zeta=1$. From Eq. (2-27), when $\zeta=1$, we have

$$c_c=2\sqrt{km} \tag{2-36}$$

Hence the damping factor ζ can be defined as

$$\zeta = \frac{c}{c_c} \tag{2-37}$$

It is a measure of the existing damping c as compared with that necessary for a system to be critically damped.

Example 4. ***Damped-Free Vibration***

A machine of 20 kg mass (44 lb_m) is mounted on springs and dampers as shown in Fig. 2-7. The total stiffness of the springs is 8 kN/m (45.7 lb_f/in.)

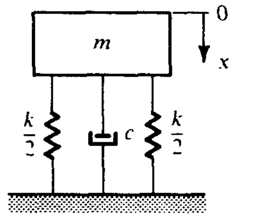


FIG. 2-7. *Damped-free vibration.*

and the total damping is 130 N·s/m (0.74 lb_f-sec/in.). If the system is initially at rest and a velocity of 100 mm/s (3.9 in./sec) is imparted to the mass, determine (a) the displacement and velocity of the mass as a time function, and (b) the displacement at $t = 1.0$ s.

Solution:

The **displacement** $x(t)$ is obtained by the direct application of Eq. (2-34). The parameters of the equation are

$$\omega_n = \sqrt{k/m} = \sqrt{8000/20} = 20 \text{ rad/s}$$

$$\zeta = c/(2m\omega_n) = 130/[2(20)(20)] = 0.1625$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 20\sqrt{1-0.1625^2} = 19.7 \text{ rad/s}$$

(a) **Substituting** these values in Eq. (2-34), we obtain

$$x = e^{-3.25t}(A_1 \cos 19.7t + A_2 \sin 19.7t)$$

$$\dot{x} = -3.25e^{-3.25t}(A_1 \cos 19.7t + A_2 \sin 19.7t)$$
$$+ 19.7e^{-3.25t}(-A_1 \sin 19.7t + A_2 \cos 19.7t)$$

Applying the initial conditions gives

$$x(0)=0 \therefore A_1=0$$

$$\dot{x}(0)=100 \text{ mm/s} \therefore A_2=100/19.7=5.07$$

$$\therefore x=5.07e^{-3.25t}\sin 19.7t \text{ mm}$$

$$\dot{x}=e^{-3.25t}(-16.47\sin 19.7t+100\cos 19.7t)$$

$$=101.3e^{-3.25t}\cos(19.7t+9.5°)\text{mm/s}$$

(b) *The displacement at* $t=1$ s *is*

$$x(t=1)=5.07e^{-3.25}\sin 19.7=0.162 \text{ mm}$$

Particular Integral

The ***particular integral*** for the excitation $F(t)=F\sin\omega t$ in Eq. *(2-18)* is of the form

$$x_p=X\sin(\omega t-\phi) \tag{2-38}$$

The values of X and ϕ can be obtained by substituting Eq. (2-38) into (2-18). This is left as an exercise. It can be shown that the **amplitude** X of the steady-state or ***harmonic response*** is

$$X=\frac{F}{\sqrt{(k-\omega^2m)^2+(\omega c)^2}} \tag{2-39}$$

$$X=\frac{F/k}{\sqrt{(1-\omega^2m/k)^2+(\omega c/k)^2}} \tag{2-40}$$

and

$$\phi=\tan^{-1}\frac{\omega c}{k-\omega^2m} \quad \text{or} \quad \phi=\tan^{-1}\frac{\omega c/k}{1-\omega^2m/k} \tag{2-41}$$

X is the amplitude of the steady-state response and $-\phi$ is the phase angle of $x_p(t)$ relative to the excitation $F\sin\omega t$, that is, the displacement lags the excitation by ϕ rad. For convenience, the last two equations are often expressed in nondimensional form. Substituting the relations $k/m=\omega_n^2$ and $\omega c/k=2\zeta\omega/\omega_n$ and defining $r=\omega/\omega_n$, these equations become

$$\frac{X}{F/k}=\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}=R \tag{2-42}$$

and

$$\phi=\tan^{-1}\frac{2\zeta r}{1-r^2} \tag{2-43}$$

where R is called the ***magnification factor*** and r the ***frequency ratio*** of the excitation frequency to the natural frequency of the system. Equations *(2-42)* and *(2-43)* are plotted in Figs. 2-8 and *2-9* with ζ as a parameter.

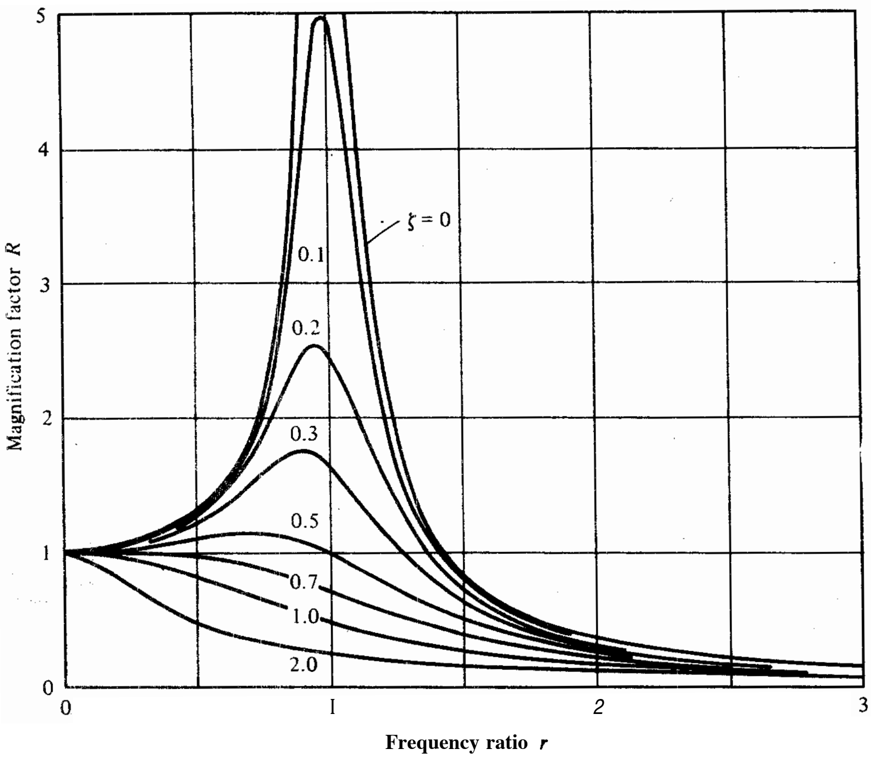


FIG. 2-8. Magnification factor-versus-frequency ratio; sysrem shown in Fig. 2-6.

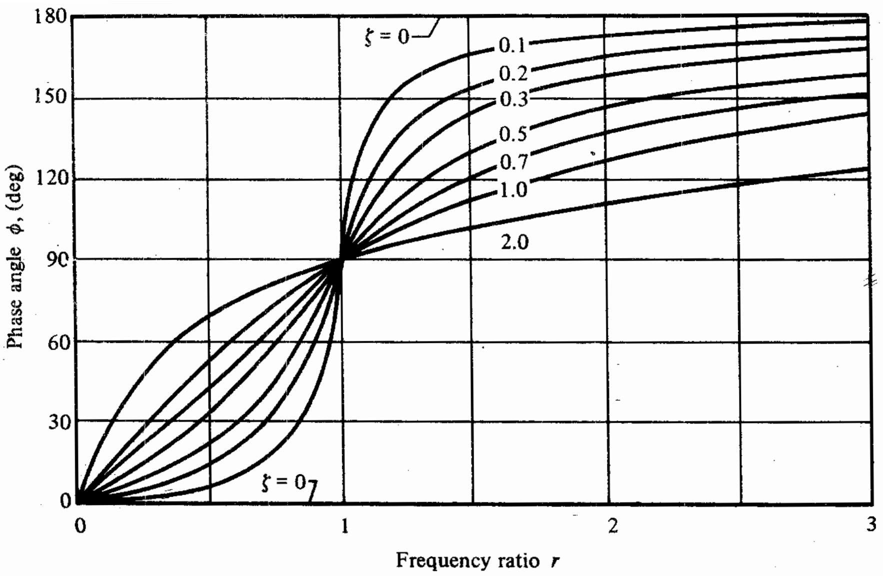


FIG. 2-9. ***Phase angle-versus-frequency ratio; system shown in Fig. 2-6.***

The characteristics of the motion $X \sin(\omega t - \phi)$ due to the excitation $F \sin \omega t$ can be observed from Eqs. (2-38) to (2-43).

1. The motion described by Eq. (2-38) is harmonic and is of the same frequency as the excitation. For a given harmonic excitation of constant amplitude F and frequency o, the amplitude X and the phase angle ϕ of the motion are constants. Hence the particular integral gives the steady-state response due to a harmonic excitation.
2. Since the particular integral does not contain arbitrary constants, the steady-state response of a system is independent of initial conditions.
3. The quantity $X/(F/k) \triangleq R$ is called the magnification factor. It is a displacement ratio, where X is the amplitude of the steady-state response and Flk is the corresponding displacement when $\omega = 0$. As shown in Fig. 2-8, R can be considerably greater than or less than unity, depending on the damping factor ζ and the frequency ratio r.
4. At resonance, when $r = \omega/\omega_n = 1$, the magnification factor R is limited only by the damping in the system. This is observed in Eq. (2-42) and Fig. 2-8 and it was explained **intuitively** in Chap. 1.
5. The phase angle ϕ as shown in Fig. 2-9 ranges from **0** to **180°**. The phase angle varies with the excitation frequency and the damping in the system. Without damping, the phase angle can only be either 0 or 180". At resonance, when $r = 1$, the phase angle is always 90"

The interpretation of phase angle can be observed from Eq. (2-38).

$$x_p = X \sin(\omega t - 4) = X \sin o(t - \phi/\omega)$$
$$= X \sin \omega(t - t_\phi)$$

where $t_\phi = \phi/\omega$ is the time shift of $x_p(t)$ relative to the excitation. In other words, the sinusoidal displacement relative to the sinusoidal excitation is shifted or delayed by an amount t_ϕ. Note that phase angle is often represented as an angle between two rotating vectors as illustrated in Fig. 1-9. Since the excitation and the response are harmonic, they can be represented by vectors as discussed in **Sec.** 1-5. However, this is an artifice, concocted for the convenience of presentation or manipulation. This will be further examined in the next section.

Several methods are commonly used to plot **Eqs.** (2-42) and (2-43). The rectangular plots in Figs. 2-8 and 2-9 are intuitive. Using the logarithmic plots in Figs. 2-10 and 2-11, it is possible to cover a wide range of frequency, such as from 10 Hz to 3,000 Hz in vibration testing. Correspondingly, the range of the magnification factor R, called the dynamic range, can be presented conveniently. The logarithmic plots also greatly facilitate the data interpretation in vibration testing.

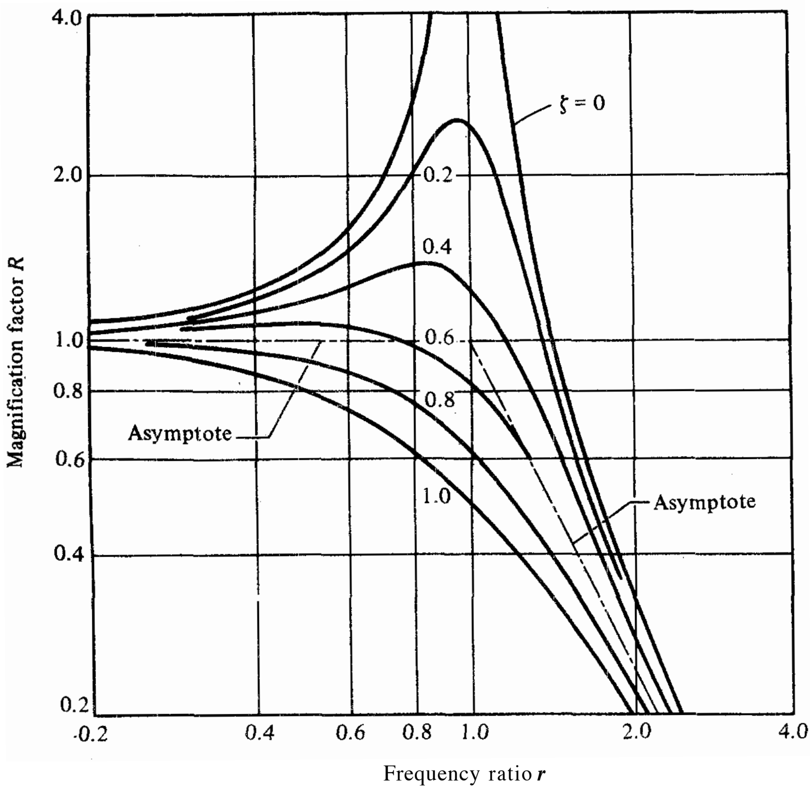


FIG. 2-10. *Magnification factor-versus-frequency ratio; system shown in Fig. 2-6.*

Another convenient method to present the steady-state response data is shown in Fig. 2-12. This format is often used to present the performance characteristics of instruments for vibration measurement. The magnification factor R as defined in Eq. (2-42) is a displacement ratio.

$$R = \frac{\text{Amplitude of steady-state displacement}}{\text{Amplitude of static displacement}} = \frac{X}{F/k} \qquad \textbf{(2-44)}$$

Similarly, the velocity ratio R_v and the acceleration ratio R_a can be defined. Since the steady-state response is $x(t) = X\sin(\omega t - \phi)$, the steady-state velocity amplitude is ωX and the acceleration amplitude is $\omega^2 X$. Dividing these quantities by ω_n and ω_n^2, respectively, the velocity ratio R_v and the acceleration ratio R_a are

$$R_v = \frac{\omega X}{\omega_n F/k} = \frac{\omega}{\omega_n} R \qquad \textbf{(2-45)}$$

$$R_a = \frac{\omega^2 X}{\omega_n^2 F/k} = \left(\frac{\omega}{\omega_n}\right)^2 R \qquad \textbf{(2-46)}$$

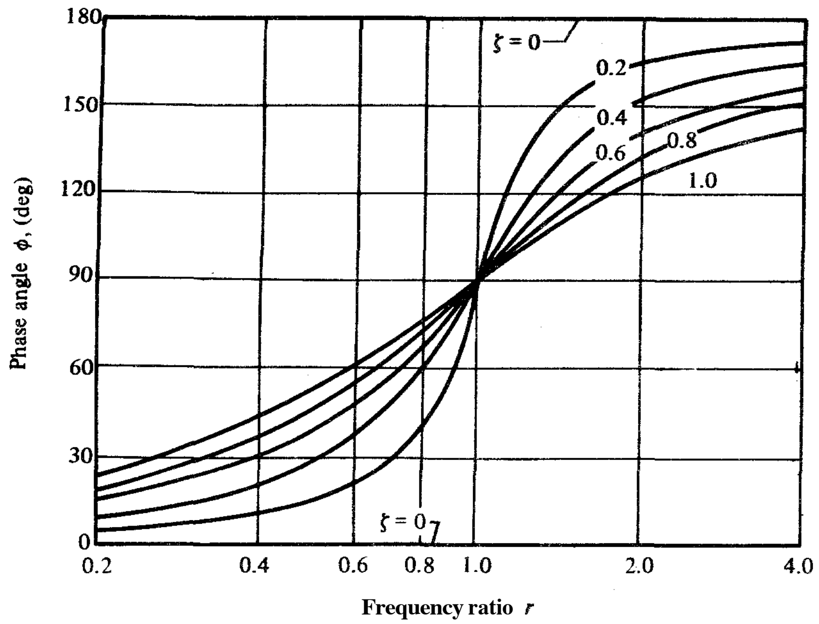


FIG. 2-11. *Phase angle-versus-frequency ratio; system shown in Fig. 2-6.*

A combined plot of R, R_v, and R_a versus the frequency ratio $r = \omega/\omega_n$ is shown in Fig. 2-12. The phase information is the same as before and needs not be presented again. At steady state, the velocity leads the displacement by 90° and the acceleration leads the velocity by 90°. The steady-state response data can be presented in other convenient forms but we shall not pursue the subject further.

General Solution

The general solution of the equation of motion in Eq. (2-18) represents the system response to a harmonic excitation and the given initial conditions. Assume that the system is underdamped, which is often encountered in vibration. Substituting Eqs. (2-35) and (2-38) into (2-20), the general solution due to a harmonic excitation is

$$x = x_c + x_p$$
$$x = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \psi) + X\sin(\omega t - \phi) \tag{2-47}$$

where X and ϕ are calculated from Eqs. (2-42) and (2-43), respectively. Note that only the constants A and ψ are arbitrary. They are evaluated by applying the initial conditions to the general solution in Eq. (2-47).

The physical interpretation of this equation was explained in Chap. 1. As the harmonic excitation and the initial conditions are applied to the

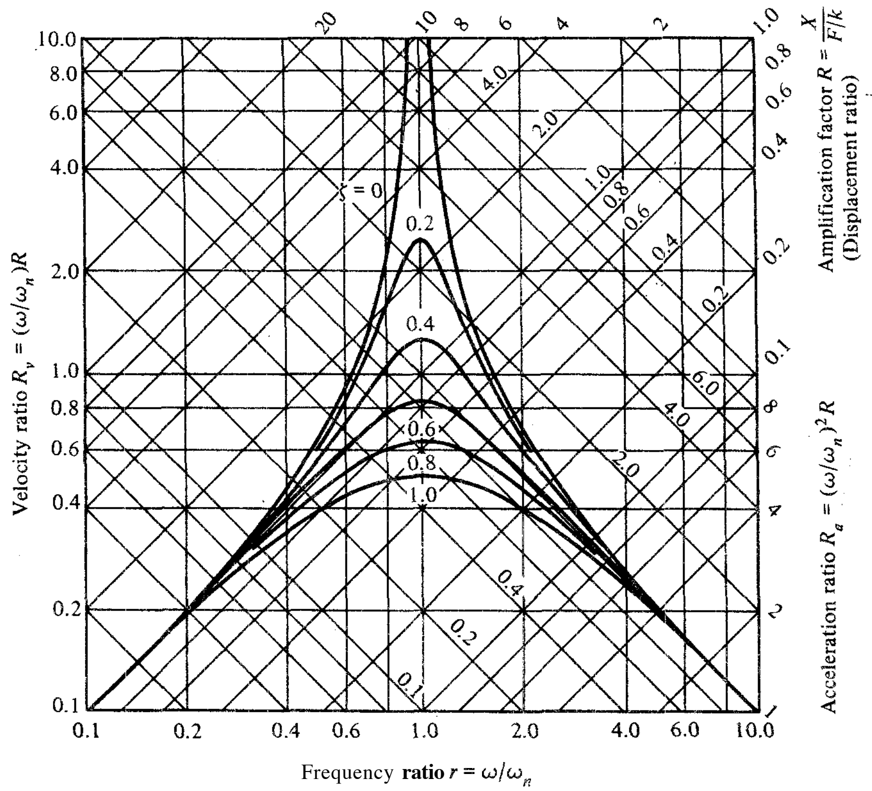


FIG. 2-12. Magnification factor, velocity ratio, and accebration ratio-versus-frequency ratio r for various damping factor ξ; system shown in Fig. 2-6.

system, it tends to follow the excitation and to vibrate at its own natural frequency. Since x_p is sustained by the excitation, it must be at the excitation frequency. On the other hand, x_c is not sustained by the excitation and it is the transient motion. The frequency ω_d of the transient motion is that of the free vibration of the system.

Example 5

Find the steady-state response and the transient motion of the system in Example *4,* if an excitation force of $24 \sin 15t$ N ($5.4 \sin 15t$ lb_f) is applied to the mass in addition to the given initial conditions.

Solution:

The displacement of the mass m is obtained *by* the direct application of Eq. (2-47). The system parameters are identical to those calculated in Example

4. The steady-state response from Eq. (2-42) is

$$x_p = \frac{F}{k} R \sin(\omega t - \phi)$$

$$= \frac{24/8{,}000}{\sqrt{[1-(15/20)^2]^2 + [2(0.1625)(15/20)]^2}} \sin(15t - \phi)$$

$$= 6.0 \sin(15t - \phi) \text{ mm}$$

where

$$\phi = \tan^{-1} \frac{2\zeta r}{1-r^2} = \tan^{-1} \frac{2(0.1625)(15/20)}{1-(15/20)^2} = 29.1^\circ$$

The general solution is

$$x = Ae^{-3.25t} \sin(19.7t + \psi) + 6.0 \sin(15t - 29.1^\circ) \text{ mm}$$

Applying the initial conditions, we obtain

$$x(0) = 0 = A \sin\psi + 6.0 \sin(-29.1^{\prime\prime})$$

$$\dot{x}(0) = 100 = A(-3.25 \sin\psi + 19.7 \cos\psi) + 6.0(15)\cos(-29.1^\circ)$$

Solving for A and ψ, we obtain $\psi = \tan^{-1} 1.87 = 61.8^{\prime\prime}$ and $A = 3.31$. Thus,

$$x = 3.31e^{-3.25t} \sin(19.7t + 61.8^\circ) + 6.0 \sin(15t - 29.1^\circ) \text{ mm}$$

The equation is plotted in Fig. 2-13.

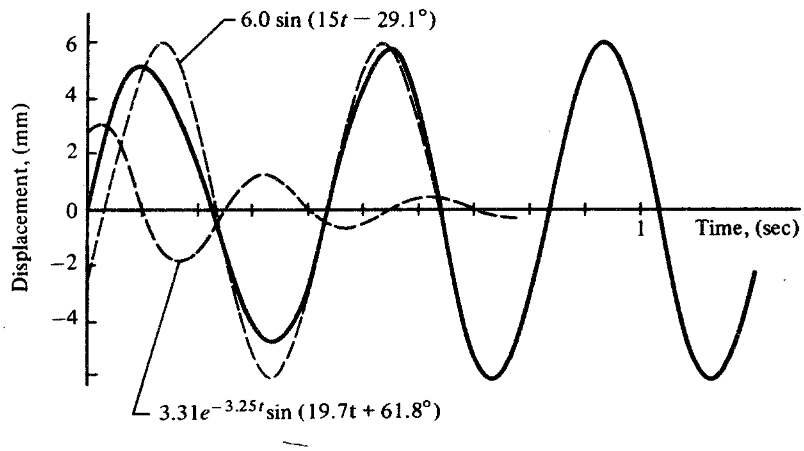


FIG. 2-13. *Displacement vs time; Example 5.*

2-6 FREQUENCY RESPONSE METHOD

Frequency response method is a harmonic analysis. A sinusoidal excitation is applied to a system and its steady-state response is examined over a frequency range of interest. For a linear system, both the excitation and the system response are sinusoidal of the same frequency. This can be verified from the theory of differential equations.

The method is generally used for vibration measurement. The implication is that it is more convenient to describe a system by its Fourier spectrum (see Chap 3). With the advent of instrumentation and computers, the pulse technique has become a popular test procedure. The results from pulse testing, however, are generally expressed as frequency response data. A vast amount of vibration measuring has been done in the past decade or two. This field of study will gain further prominence in the future.*

We shall discuss mechanical impedance method and sinusoidal transfer function in this section.

Impedance Method

Mechanical impedance method is a harmonic analysis. It represents the sinusoidal functions in the equation of motion by means of rotating vectors as discussed in Sec. 1-5. We shall first represent the forces in a system by means of rotating vectors and then derive the mechanical impedance of the system and its components.

The equation of motion of the one-degree-of-freedom system in Fig. 2-6 and its steady-state response from Eq. (2-38) are

$$m\ddot{x} + c\dot{x} + kx = F \sin \omega t$$
$$x = X \sin(\omega t - \phi) \tag{2-48}$$

Using the vectorial representation of hartnonic motions, the equations above can be expressed as

$$m\ddot{x} + c\dot{x} + kx = \bar{F}e^{j\omega t} \tag{2-49}$$

$$x = \bar{X}e^{j\omega t} \tag{2-50}$$

*** The importance of vibration measurement can be judged from a quotation: "Mechanical Maintenance is one of the largest industries on earth. Vibration measuring has helped to lower costs by as much as 50%." M. P. Blake, Monograph #721111, Lovejoy, Inc., Downers Grove, Ill., Dec. 1972.**

For discussion of vibration measurement, see, for example, M. P. Blake and Wm. S. Mitchell, *Vibration and Acoustic Measurement Handbook,* Spartan Books, New York, 1972

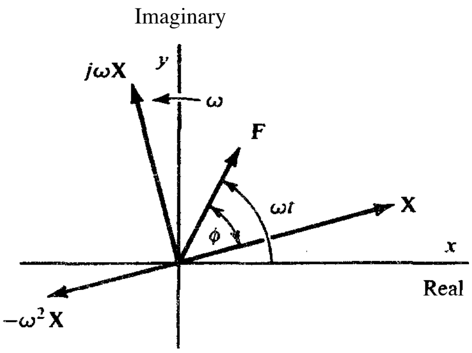


FIG. 2-14. *Displacement, velocity and acceleration vectors.*

The force vector is $\mathbf{F} = \bar{F}e^{j\omega t}$ and the displacement vector is $\mathbf{X} = \bar{X}e^{j\omega t}$. $\bar{F}$ and X are the phasors of **F** and X, respectively.*

The force vector **F** and the displacement vector **X** are-shown in Fig. 2-14. The corresponding velocity and acceleration vectors are obtained from X by differentiation as shown in Eq. (1-10). The velocity vector is $j\omega\mathbf{X}$ and the acceleration vector is $-\mathbf{w}^2\mathbf{X}$. The relative positions of the vectors are illustrated in the figure.

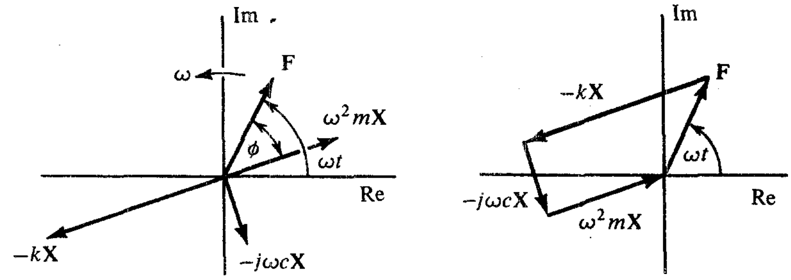
The harmonic forces are obtained by multiplying each of these vectors by the appropriate constants. The spring force always resists the displacement. Hence the spring force vector is $-k\mathbf{X}$. Similarly, the damping and the inertia-force vectors are $-j\omega c\mathbf{X}$ and $\omega^2 m\mathbf{X}$, respectively. These vectors are shown in Fig. 2-15(*a*). Although these are rotating vectors, their relative positions or phase angles are constant. For dynamic equilibrium, the vectorial sum of the forces due to the spring, the damper, and the mass is equal and opposite to the applied force as indicated in Eq. (2-49). Hence the force vectors form a closed polygon as shown in Fig. 2-15(*b*).

Figure 2-16 shows the relation of these vectors for an excitation force of constant magnitude but for frequency ratio $r \gtreqless 1$, where $r = \omega/\omega_n$. The

*The phasor notation is often a source of confusion for some students. A phasor is a time independent complex coefficient, which together with the factor $e^{j\omega t}$ gives a complex time function.

From Sec. 1-5, a phasor is a complex amplitude or a complex number. It denotes the magnitude and phase angle of a vector relative to the reference vector. In this case, the force is the reference vector and its phase angle is zero. Thus, $\bar{F} = F$ or $\bar{F} = Fe^{j\alpha}$ where a = 0. If given $F(t) = F\sin\omega t$ and $x = X\sin(\omega t - \phi)$, the displacement vector is $X = Xe^{j(\omega t - \phi)} = \bar{X}eJ^{w'}$. Hence the phasor of X is $Xe^{-j\phi}$.

More generally, if given $F(t) = F\sin(\omega t + \beta)$, the steady-state response is $x =$ X $\sin(\omega t + \beta - 4)$. In phasor notation. the force vector is $F = Fe^{j(\omega t + \beta)} = \bar{F}e^{j\omega t}$ and the displacement vector is $X = Xe^{j(\omega t - \beta - \phi)} = \bar{X}e^{j\omega t}$. Hence the phasor of **F** is $Fe^{j\beta}$ and that of X is $Xe^{j(\beta - \phi)}$. The relative amplitude and the phase angle between the force and the displacement remain unchanged.



(*a*) Force vectors (*b*) **Polygon** of force vectors

FIG. 2-15. ***Vectors representing harmonic spring, damping, inertial, and excitation forces.***

magnitudes and the phase angles indicated may be compared with those shown in Figs. **2-8** and **2-9.** Since the interest is in the relative amplitudes and phase angles of the vectors, the vectors are rotated clockwise by an amount $(\omega t - \phi)$. This is equivalent to choosing $(\omega t - \phi)$ as the datum of measurement.

The **displacement** $x(t)$ is obtained by substituting Eq. (2-50) in (2-49). Factoring out the $e^{j\omega t}$ term, we get

$$[(j\omega)^2 m + (j\omega)c + k]\bar{X} = \bar{F} \tag{2-51}$$

or

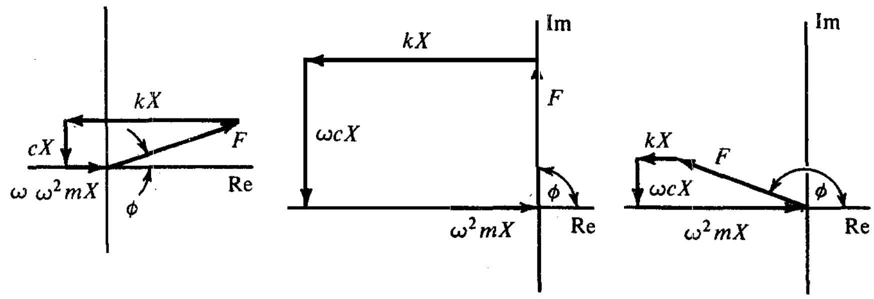
$$\bar{X} = \frac{\bar{F}}{k - \omega^2 m + j\omega c} = Xe^{-j\phi} \tag{2-52}$$

where

$$X = |\bar{X}| = \frac{F}{\sqrt{(k-\omega^2 m)^2 + (\omega c)^2}} = \frac{F}{k} R \tag{2-53}$$

$$-\phi = \angle\bar{X} = -\tan^{-1}\frac{\omega c}{k - \omega^2 m} \tag{2-54}$$

where R is the magnification factor defined in Eq. **(2-42).**



(*a*) $r = 0.5$ (*b*) $r = 1.0$ (*c*) $r = 2.0$

FIG. 2-16. Polygon of force vectors for frequency ratio $r \gtrless 1$: F = constant and $\zeta = 0.25$.

TABLE 2-1. Impedance of *System Elements*

ELEMENT	SYMBOL	IMPEDANCE
Mass	m	$--w^2m$
Damper	c	$j\omega c$
Spring	k	k

The quantity $(k - w^2m + j\omega c)$ in Eq. (2-51) is called 'the ***mechanical impedance*** of the system in Fig. 2-6. It has the dimension of ***force per unit displacement.*** The definition follows conveniently from Newton's law of motion in Eq. (2-51). Similarly, for the elements m, c, and **k**, the corresponding mechanical impedances are defined as -0^2m, $j\omega c$, and **k**, respectively. These are tabulated in Table 2-1. In the literature, mechanical impedance is also defined as force per unit velocity, although this definition is by no means standardized.*

*Mechanical impedance was defined by analogy from Ohm's law. Let us briefly examine the analogy.

Electrical impedance was defined from the generalization of Ohm's law $V = RI$, where V is the voltage drop across the resistor R and I the current flow through R. The generalization is $V = ZI$, where Z is the impedance of a component or a network.

Let us rewrite Eq. (2-49) and compare it with the ***RLC*** circuit in series and in parallel shown in Fig. 2-17.

$$m\frac{d\dot{x}}{dt} + c\dot{x} + k\int \dot{x}\,dt = F(t)$$

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v(t)$$

$$C\frac{dv}{dt} + \frac{1}{R}v + \frac{1}{L}\int v\,dt = i(t)$$

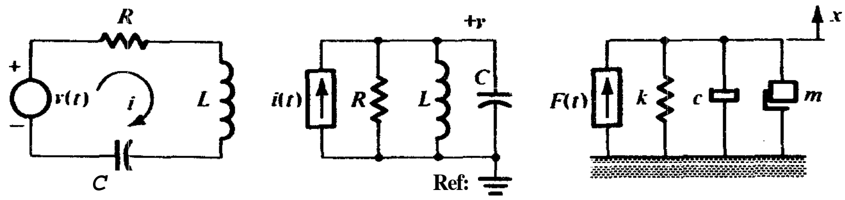
where $F(t)$, $v(t)$, and $i(t)$ are the sources of force, voltage, and current, respectively. Since all the equations above are of the same form, either the force-voltage analogy or the force-current analogy can be used to define mechanical impedance.

Using the force-voltage analogy, mechanical impedance is defined as **force/velocity.** Using the force-current analogy, mechanical impedance is defined as **velocity/force.**

The electrical circuits in Fig. 2-17 are self-explanatory. The mechanical "circuit" shows that (1) the excitation force is the sum of the inertia force, the damping force, and the spring force, and (2) the mass, spring, and damper have the same velocity at a common junction. Hence the diagram represents the mechanical system.

Further examination of the diagrams reveals that, if the force-current analogy is used, the mechanical circuit can be obtained directly from the electrical. In other words, if (1) force is analogous to current, (2) velocity analogous to voltage, and mechanical impedance defined as **velocity/force,** then both the equations and the circuits are analogous.

The force-voltage analogy is intuitive. 'The force-current analogy has the advantages mentioned above. Furthermore, a force acts through a component; the forces at both ends of a spring are equal. Note that a current flows through a component. Hence force and current are both through variables. The velocity is measured across a component. Note also that a voltage is measured across a component. Thus, velocity and voltage are both across variables. Using this concept and the force-current analogy, the electrical and mechanical circuits should be alike.



(a) RLC in series (b) PLC in parallel (c) Mechanical circuit

FIG. 2-17. *Comparison of electrical and mechanical circuits.*

Example 6

Use the impedance method to find the steady-state response of the system described in Example 5.

Solution:

The impedance of the system is

$$(k - 0^2 m) + j\omega c = [8000 - 15^2(20)] + j15(130)$$
$$= 3500 + j1950 = 4007\angle 29.1^\circ$$

From Eq. *(2-52)* we have

$$Xe^{-j\phi} = \frac{24}{4007\angle 29.1^\circ} = 0.006\angle -29.1^\circ$$

which is the phasor or complex amplitude of the displacement vector X.

$$\mathbf{X} = Xe^{j(\omega t - \phi)} = 0.006e^{j(15t - 29.1^\circ)}$$

Since the given excitation force is $F \sin \omega t$, which is equal to $\text{Im}(Fe^{j\omega t})$, the displacement $x(t)$ is $\text{Im}[Xe^{j(\omega t - \phi)}]$, which is $X \sin(\omega t - \phi)$.

$$x = 6.0 \sin(15t - 29.1^\circ) \text{ mm}$$

Transfer Function

The transfer function is a mathematical model defining the input-output relation of a physical system. If the system has a single input and a single output, it can be represented by means of a block diagram shown in Fig. 2-18. The system response $x(t)$ is caused by an excitation $F(t)$. Naming

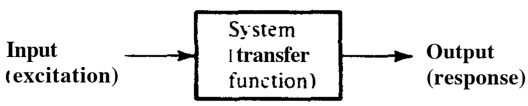


FIG. 2-18. *Block* diagram of linear *systems.*

$x(t)$ as the output and $F(t)$ the input, this *causal relation* is specified by the transfer function.

$$(\text{Output}) = (\text{transfer function})(\text{input}) \tag{2-55}$$

or

$$\frac{(\text{Output})}{(\text{Input})} = (\text{transfer function}) \tag{2-56}$$

Consider the system in Fig. 2-6 as an example. The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

Using the impedance method and substituting $j\omega$ for the time derivative in the equation, we obtain

$$\frac{x}{F}(j\omega) = \frac{1}{(k - \omega^2 m) + j\omega c} = G(j\omega) \tag{2-57}$$

The symbol $G(j\omega)$ indicates that G is a function of ω. Similarly, x/F is a function of ω, that is, the symbol does *not* indicate a product of (x/F) and $(j\omega)$. $G(j\omega)$ is the sinusoidal transfer function of the system.

Comparing Eqs. (2-57) and (2-52), it is evident that the transfer function is another technique to present the frequency response data of a system. Moreover, the data in Figs. 2-8 to 2-12 are the nondimensionai plots of the sinusoidal transfer function. The transfer function of a complex system can be obtained from test data. Thus, a system can be identified by data from its frequency response test.

Note that the transfer function defined in Eq. (2-55) is an operator. It operates on the input to yield an output. It is often called a ratio of output per unit input. This is not a ratio in the normal sense of the word as a ratio of two numbers. As illustrated in Eq. (2-57), the transfer function is a complex number. Furthermore, it is dimensional, such as displacement per unit force. It is more appropriate to think of a transfer function simply as an operator.

Example 7

Determine the frequency response of the system described in Example 5 by means of its transfer function.

Solution:

From Eq. (2-57) **the transfer function is**

$$G(j\omega) = \frac{1}{(k - \omega^2 m) + j\omega c}$$

$$= \frac{1}{[8000 - 15^2(20)] + j15(130)} = 0.25 \times 10^{-3} \underline{/-29.1^\circ}$$

Hence $\frac{x}{F}(j\omega) = 0.25 \times 10^{-3}\underline{/29.1^\circ}$. **Since the excitation is 24 sin** $15t$, **the magnitude of the displacement is**

$$X = 24(0.25 \times 10^{-3}) = 0.006 \text{ m}$$

or

$$x = 6.0 \sin(15t - 29.1^\circ) \text{ mm}$$

Resonance, Damping, and Bandwidth

It is observed in Fig. 2-8 that the height of the resonance peak is a function of the damping in the system. One of these frequency response curves is reproduced in Fig. 2-19. It can be shown that the peak of the resonance curves occur at $r = \sqrt{1 - 2\zeta^2}$. If $\zeta \leq 0.1$, the peaks occur at $r \simeq 1$. Thus, from Eq. **(2-42)** the value of the maximum amplification factor is

$$R_{max} \simeq \frac{1}{2\zeta} \tag{2-58}$$

The damping in a system is indicated by the sharpness of its response curve near resonance and can be measured by the ***bandwidth.*** The bandwidth is $(r_2 - r_1)$ as shown in Fig. 2-19, where $r = \omega/\omega_n$ is a frequency ratio and r_1 and r_2 are the frequency ratios at the ***half-power points.**** The amplification factor R at r_1 and r_2 is $R = R_{max}/\sqrt{2}$. Substituting this in Eq. (2-42) and letting R_{max} = **1/25** shown *in* **Eq.** (2-58), we obtain

$$\left(\frac{1}{2\zeta}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

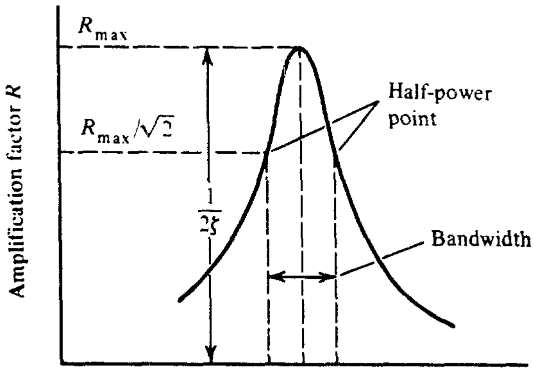


FIG. 2-19. ***Harmonic** response **curve** showing bandwidth and half-power points.*

* **This terminology is commonly used in electrical engineering. The power dissipation P in a resistor R is** $P_1 = I_1^2R$. *At* **half power,** $P_2 = P_1/2 = I_2^2R$. **Thus,** $I_2 = I_1/\sqrt{2}$.

Assuming $\zeta \le 0.1$ and solving for r, we get

$$r_{1,2} \simeq 1 \pm \zeta \tag{2-59}$$

$$\text{Bandwidth} = r_2 - r_1 = \frac{\omega_2}{\omega_n} - \frac{\omega_1}{\omega_n} = 2\zeta \tag{2-60}$$

A factor Q is also used to define the bandwidth and damping.

$$Q = \frac{1}{\text{bandwidth}} = \frac{1}{2\zeta} \tag{2-61}$$

Q is used to measure the *quality* of a resonance circuit in electrical engineering. It is also useful for determining the equivalent viscous damping in a mechanical system.

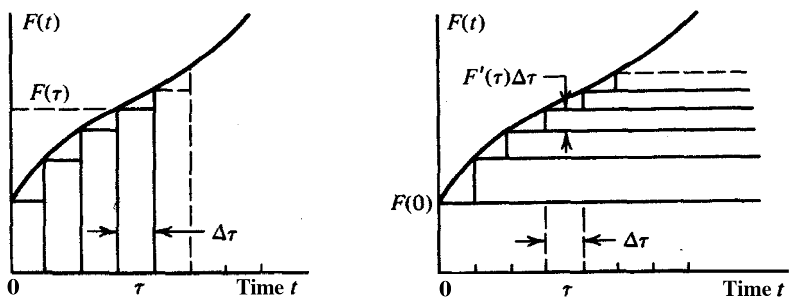
2-7 TRANSIENT VIBRATION

We shall show that the transient vibration due to an arbitrary excitation $F(t)$ can be obtained by means of superposition. Although the method is not convenient for hand calculations, it can be implemented readily using computers.

The equation of motion of the model in Fig. 2-6 for systems with one degree of freedom and an excitation $F(t)$ is

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{2-62}$$

One method to solve the equation is to approximate $F(t)$ by a sequence of pulses as shown in Fig. 2-20(*a*). If the system response to a typical pulse input is known, the response to $F(t)$ can be obtained by superposition. In other words, the system response to $F(t)$ is the sum of the responses due to each of the pulses in the sequence.



(*a*) Sequence of pulses (b) Sequence of steps

FIG. 2-20. $F(t)$ approximated by pulses and steps.

Impulse Response

The system response due to a unit impulse input with zero initial conditions is called its *impulse response.* A rectangular pulse of duration or width T_0 and height $1/T_0$ is shown in Fig. 2-21(*a*). The area of this pulse is unity. To obtain a unit *impulse,* let the pulse width T_0 approach zero while maintaining the pulse area at unity. In the limit, we have a unit impulse $\delta(t)$ as defined by the relations

$$\begin{aligned} \delta(t) &= 0 \quad \text{for} \quad t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)\, dt &= 1 \end{aligned} \tag{2-63}$$

This impulse occurs at $t = 0$ as shown in Fig. 2-21(*b*). If a unit impulse occurs at $t = \tau$, it is defined by the relations

$$\begin{aligned} \delta(t-\tau) &= 0 \quad \text{for} \quad t \neq \tau \\ \int_{-\infty}^{\infty} \delta(t-\tau)\, dt &= 1 \end{aligned} \tag{2-64}$$

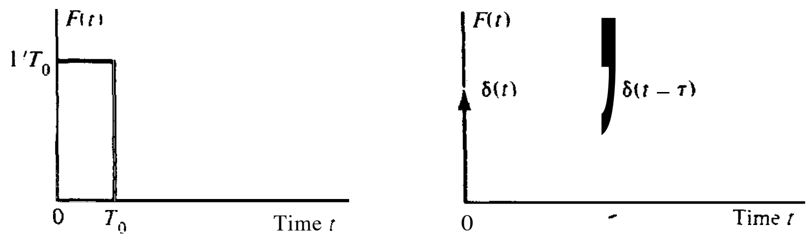
Note that $\delta(t-\tau)$ is a unit impulse translated along the positive time axis by an amount τ.

Any function, not necessarily a rectangular pulse. having the properties above can be used as a unit impulse and is called the *Dirac* delta *function.* Mathematically. a unit impulse must have zero pulse width. unit area, and infinite height. It seems that an impulse cannot be realized in applications. In pulse testing of real systems, however. an excitation can be considered as an impulse if its duration is very short compared with the natural period ($= 1/f_n$) of the system.

From Eq. (2-62), the equation of motion with an excitation $F(t) = \delta(t)$ is

$$m\ddot{x} + c\dot{x} + kx = \delta(t) \tag{2-65}$$

Assume that the system is at rest before the unit impulse: $\delta(t)$ is applied,



(*a*) Rectangular pulse with unit area (*b*) Unit impulse $\delta(t)$ and $\delta(t-\tau)$

FIG. 2-21. Rectangular puke and unit *impulse.*

that is,

$$x(0^-) = \dot{x}(0^-) = 0 \tag{2-66}$$

Since $\delta(t)$ is applied at $t=0$, it is over with at $t \gtrsim 0^+$. Thus, *(1)* the system becomes unforced for $t \gtrsim 0^+$, and *(2)* the energy input due to $\delta(t)$ becomes the initial conditions at $t = 0^+$.

To find the initial conditions at $t = 0^+$, we integrate **Eq.** *(2-65)* twice for $0^- \leq t \leq 0^+$. Thus,

$$m[x(0^+) - x(0^-)] + \int_{0^-}^{0^+} cx\,dt + \iint_{0^-}^{0^+} kx\ dt\,dt = \iint_{0^-}^{0^+} \delta(t)\ dt\,dt$$

From **Eq.** (2-63), the first integration of $\delta(t)$ gives a constant and the second integration for $0^- \leq t \leq 0^+$ is zero. Hence the right side of the equation is zero. If $x(t)$ does not become infinite, its integration over this infinitesimal interval is also zero. Thus,

$$m[x(0^+) - x(0^-)] + 0 + 0 = 0$$

If $x(0^-) = 0$ as indicated in Eq. (2-66), we have $x(0^+) = 0$.

Now, integrating **Eq.** *(2-65)* once for $0^- \leq t \leq 0^+$ gives

$$m[\dot{x}(0^+) - \dot{x}(0^-)] + c[x(0^+) - x(0^-)] + \int_{0^-}^{0^+} kx\ dt = \int_{0^-}^{0^-} \delta(t)\ dt$$

From **Eq.** (2-63), the right side of this equation is unity. The third term on the left side is zero if $x(t)$ does not become infinite. The second term is zero as explained above. Thus,

$$m[\dot{x}(0^+) - \dot{x}(0^-)] + 0 + 0 = 1$$

For $x(0^-) = \dot{x}(0^-) = 0$ in **Eq.** (2-66), the initial conditions at $t = 0$ due to a unit impulse at $t = 0$ are

$$x(0^+) = 0 \qquad \text{and} \qquad \dot{x}(0^+) = 1/m \tag{2-67}$$

The homogeneous equation equivalent to Eq. *(2-65)* is

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2-68}$$

with the initial conditions $x(0^+) = 0$ and $\dot{x}(0^+) = 1/m$. This deduction is almost intuitive, since an impulse would cause a momentum change. If m is constant and $x(0^-) = \dot{x}(0^-) = 0$, an impulse would cause a change in the initial velocity.

It can be shown from the solution of Eq. *(2-68)* that the impulse response $h(t)$ is

$$h(t) = \frac{1}{\omega_d m} e^{-\zeta\omega_n t} \sin \omega_d t, \qquad \text{for} \qquad t > 0 \tag{2-69}$$

If a unit impulse $\delta(t - \tau)$ occurs at $t = \tau$ as shown in Fig. 2-21(b), the response is delayed by an amount τ, that is,

$$h(t-\tau)=\frac{1}{\omega_d m}e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\qquad\text{for } t>\tau \tag{2-70}$$

where $h(t-\tau)=0$ for $t<\tau$.

Convolution Integral

Let an excitation $F(t)$ be approximated by a sequence of pulses as shown in Fig. 2-20(*a*). The strength of a pulse is defined by the pulse area. The strength of a typical pulse in the sequence at time τ is the area $F(\tau)\Delta\tau$. The system response to a typical pulse is the product of its unit impulse response and the pulse strength, that is, $h(t-\tau)[F(\tau)\Delta\tau]$. By superposition, we sum the responses due to each of the pulses in the sequence and obtain

$$x(t)=\sum h(t-\tau)F(\tau)\Delta\tau$$

As $\Delta\tau$ approaches zero, the summation becomes the convolution integral

$$x(t)=\int_0^t F(\tau)h(t-\tau)\,d\tau=\frac{1}{\omega_d m}\int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau \tag{2-71}$$

This is the system response for the input $F(t)$ with zero initial conditions. An alternative form of the integral is

$$x(t)=\int_0^t F(t-\tau)h(\tau)\,d\tau \tag{2-72}$$

In other words, the response of a linear system to an arbitrary excitation is the convolution of its impulse response and the excitation. This statement is known as Borel's theorem.

If the initial conditions are not zero, the complete solution is obtained by the superposition of the particular solution due to the excitation and the complementary solution due to the initial conditions. Substituting the initial conditions $x(0)=x_0$ and $\dot{x}(0)=\dot{x}_0$ into Eq. (2-34) gives the complementary solution

$$x=e^{-\zeta\omega_n t}\left(x_0\cos\omega_d t+\frac{\dot{x}_0+\zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right) \tag{2-73}$$

The particular solution is shown in Eq. (2-71). By superposition, the complete solution is

$$x=e^{-\zeta\omega_n t}\left(x_0\cos\omega_d t+\frac{\dot{x}_0+\zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right)+\int_0^t F(\tau)h(t-\tau)\,d\tau \tag{2-74}$$

The convolution integral is a powerful tool in the study of linear systems. Although Eq. (2–71) cannot be conveniently applied by hand calculations, it can be implemented readily using computers. The example to follow is not indicative of the amount of algebraic computation involved in applying the convolution integral by hand calculations.

Example **8***

A box shown in Fig. *2-22* is dropped through a height H. Find the maximum force transmitted to the body m when the box strikes the floor. Assume there is sufficient clearance between m and the box to avoid contact.

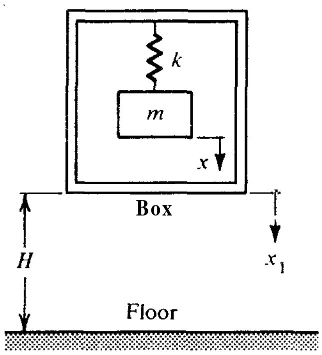


FIG. *2-22.* Drop *test.*

Solution:

Let $x(t)$ be the relative distance between m and the box and $t_0=\sqrt{2H/g}$ the time for the box to strike the floor. Assume that on striking the floor the box remains in contact with the floor. Let us consider the time interval for the free fall and that after striking the floor separately.

During the free fall, the absolute displacement of m is $(x+x_1)$. Hence the equation of motion of m is

$$m(\ddot{x}+\ddot{x}_1)=-kx \qquad \text{or} \qquad m\ddot{x}+kx=-m\ddot{x}_1$$

where

$$x_1=\tfrac{1}{2}gt^2 \qquad \text{or} \qquad \ddot{x}_1=g$$

Hence the equation of motion becomes

$$\ddot{x}+\omega_n^2x=-g$$

If the box is initially at rest before the fall, we have zero initial conditions. Applying Eq. *(2-74)* where $F(t)=-g$ we get

$$x=0+\int_0^t(-g)h(t-\tau)\,d\tau$$

*R. D. Mindlin, "Dynamics of Package Cushioning," Bell *Syst. Tech. Jour.,* 24, (July 1945) pp. 353–461.

The expression for $h(t-\tau)$ is obtained from Eq. (2-70). Since the system is undamped, the equation above becomes

$$x = -\frac{g}{\omega_n}\int_0^t \sin\omega_n(t-\tau)\,d\tau$$

$$= -\frac{g}{\omega_n^2}(1-\cos\omega_n t)$$

From the time that the box strikes the floor at $t=t_0$, the system becomes unforced. Redefining the time from the instant of impact, the initial conditions are

$$x(0)=x(t_0)=-\frac{g}{\omega_n^2}(1-\cos\omega_n t_0)$$

$$\dot{x}(0)=\dot{x}(t_0)+gt_0=gt_0-\frac{g}{\omega_n}\sin\omega_n t_0$$

where $x(t_0)$ and $\dot{x}(t_0)$ are obtained from the $x(t)$ above and gt_0 is the velocity of the box assembly at $t=t_0$. Applying Eq. (2-7.4) with $F(t)=0$ gives

$$x = -\frac{g}{\omega_n^2}(1-\cos\omega_n t_0)\cos\omega_n t+\left(\frac{gt_0}{\omega_n}-\frac{g}{\omega_n^2}\sin\omega_n t_0\right)\sin\omega_n t$$

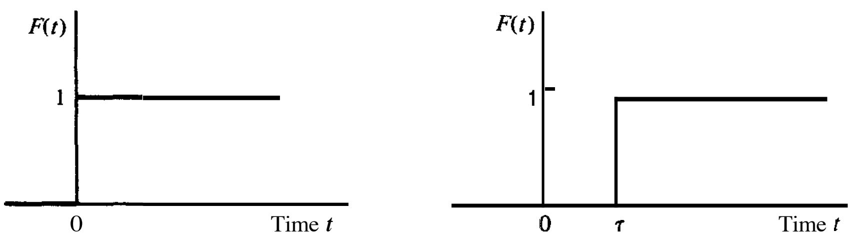
The maximum force transmitted to m is $m(\ddot{x}_{max})=kX$, where X is the amplitude of $x(t)$. Thus, the maximum force is

$$\text{Force}=\frac{kg}{\omega_n^2}\sqrt{(1-\cos\omega_n t_0)^2+(\omega_n t_0-\sin\omega_n t_0)^2}$$

Indicial Response

The system response due to a unit step input with zero initial conditions is called the *indicial response.* A *unit step function* $u(t)$ shown in Fig. 2-23(a) has the property

$$u(t)=\begin{cases}1 & \text{for } t>0\\ 0 & \text{for } t<0\end{cases} \qquad \textbf{(2-75)}$$



(a) Unit step function $u(t)$ (b) Unit step function $u(t-\tau)$

Fig. 2-23. Unit step functions.

A unit step translated along the positive time axis by an amount τ to become $u(t-\tau)$ is shown in Fig. 2-22(*b*).

$$u(t-\tau)=\begin{cases}1 & \text{for } t>\tau \\ 0 & \text{for } t<\tau\end{cases} \tag{2-76}$$

An arbitrary function $F(t)$ can alternatively be approximated by a sequence of steps as illustrated in Fig. 2-20(*b*). Following the steps enumerated for the impulse response, it can be shown that the **indicial** response $x_u(t)$ is

$$x_u(t)=\frac{1}{k'}\left[1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)\right] \tag{2-77}$$

where 4 = sin-' $\sqrt{1-\zeta^2}$. The system response to an arbitrary input $F(t)$ is the superposition of the responses due to each of the individual steps. Thus,

$$x(t)=F(0)x_u(t)+\int_0^t F'(\tau)x_u(t-\tau)\,d\tau \tag{2-78}$$

where $F'(\tau)$ is the time derivative of $F(t)$ estimated at $t=\tau$. The term $F(0)x_u(t)$ is to account for the step at $t=0$, since the slope does not take into account $F(0)$. Equation (2-78) is referred to as ***Duhamel's integral*** or the ***superposition integral.***

2-8 COMPARISON OF RECTILINEAR AND ROTATIONAL SYSTEMS

The discussions in the previous sections centered on systems with rectilinear motion. The theory and the interpretations given are equally applicable to systems with rotational motion. The analogy between the two types of motion and the units normally employed are tabulated in Table 2-2. The responses of the two types of systems are compared in Table *2-3*.

Extending this analogy concept, it may be said that systems are analogous if they are described by equations of the same form. The theory developed for one system is applicable to its analogous systems. It will be shown in the next chapter that our study of the generalized model for systems with one degree of freedom is applicable to a large number of physical problems, the appearance of which may bear little resemblance to one another.

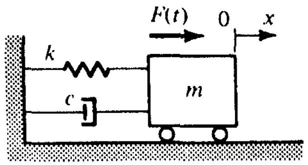
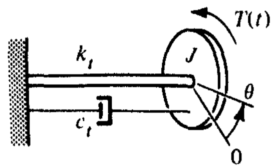
2-9 SUMMARY

The number of degrees of freedom of a system is the number of spatial coordinates required to specify its configuration minus the number of

TABLE 2-2. *Analogy between Rectilinear and Rotational Systems*

QUANTITY	RECTILINEAR SYSTEM			ROTATIONAL SYSTEM		
	SYMBOL	UNIT		SYMBOL	UNIT	
		ENGLISH	SI UNIT		ENGLISH	SI UNIT
Time	t	sec	s	t	sec	s
Displacement	x	in.	m	8	rad	rad
Velocity	$\dot{x}$	in./sec	m/s	$\dot{\theta}$	rad/sec	rad/s
Acceleration	$\ddot{x}$	in./sec^2	m/s^2	$\ddot{\theta}$	rad/sec^2	rad/s^2
Mass, moment of inertia	m	lb$_f$-sec^2/in.	kg	J	in.-lb$_f$-sec^2	m^2 . kg
Damping factor	c	lb$_f$-sec/in.	s · N/m	c_t	in.lb$_f$-sec/rad	m · s . Nlrad
Spring constant	k	lb$_f$/in.	N/m	k_t	in.-lb$_f$/rad	m · N/rad
Force, torque	$F = m\ddot{x}$	lb$_f$	N = m · kg/s^2	$T = J\ddot{\theta}$	in.-lb$_f$	m · N = m^2 · kg/s^2
Momentum	$m\dot{x}$	lb$_f$-sec	s · N = m · kg/s	$J\dot{\theta}$	in.-lb$_f$-sec	m^2 · kg . rad/s
Impulse	Ft	lb$_f$-sec	s · N	Tt	in.-lb$_f$-sec	m^2 . kg · rad/s
Kinetic energy	$T = \frac{1}{2}m\dot{x}^2$	in.-lb$_f$	J	$T = \frac{1}{2}J\dot{\theta}^2$	in.-lb$_f$	J
Potential energy	$U = \frac{1}{2}kx^2$	in.-lb$_f$	J	$U = \frac{1}{2}k_t\theta^2$	in.-lb$_f$	J
Work	$\int F\,dx$	in.-lb$_f$	J = m · N = m^2 · kg/s^2	$\int T\,d\theta$	in.-lb$_f$	J = m · N = m^2 · kg/s^2
Natural frequency	$\omega_n = \sqrt{k/m}$	rad/sec	rad/s	$\omega_n = \sqrt{k_t/J}$	rad/sec	rad/s
	$f_n = \frac{\omega_n}{2\pi}$	Hz	Hz	$f_n = \frac{\omega_n}{2\pi}$	Hz	Hz

Table 2-3. *Response of Rectilinear and Rotational Systems*

ITEM	RECTILINEAR SYSTEM	ROTATIONAL SYSTEM
System		
Equation of motion	$m\ddot{x} + c\dot{x} + kx = F(t)$	$J\ddot{\theta} + c_t\dot{\theta} + k_t\theta = T(t)$
System response	$x = x_c + x_p$	$\theta = \theta_c + \theta_p$
Initial conditions	$x(0) = x_0,\ \dot{x}(0) = \dot{x}_0$	$\theta(0) = \theta_0,\ \dot{\theta}(0) = \dot{\theta}_0$
Complementary function, $x_c(t)$	$x_c = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \psi)$ $\omega_n^2 = k/m,\ \zeta = \frac{1}{2}c/\sqrt{km},\ \omega_d = \sqrt{1-\zeta^2}\,\omega_n$	$\theta_c = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \psi)$ $\omega_n^2 = k_t/J,\ \zeta = \frac{1}{2}c_t/\sqrt{k_t J},\ \omega_d = \sqrt{1-\zeta^2}\,\omega_n$
Particular integral, $x_p(t)$		
(1) $F(t) = F\sin wt$	$x_p = X\sin(\omega t - \phi)$	$\theta_p = \Theta\sin(\omega t - \phi)$
	$X = \frac{F}{k}R = \frac{F/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$	$\Theta = \frac{T}{k_t}R = \frac{T/k_t}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$
	$r = \omega/\omega_n,\ \phi = \tan^{-1} 2\zeta r/(1-r^2)$	$r = \omega/\omega_n,\ \phi = \tan^{-1} 2\zeta r/(1-r^2)$
(2) $F(t) = \delta(t)$	$h(t) = \frac{1}{\omega_d m}e^{-\zeta\omega_n t}\sin\omega_d t$	$h(t) = \frac{1}{\omega_d J}e^{-\zeta\omega_n t}\sin\omega_d t$
(3) $F(t)$ arbitrary	$x_p = \int_0^t F(\tau)h(t-\tau)\,d\tau$	$\theta_p = \int_0^t T(\tau)h(t-\tau)\,d\tau$

equations of constraint. Many practical problems can be represented by systems with one degree of freedom, the model of which is shown in Fig. 2-6.

The methods of study in this chapter can be broadly classified as the time and the frequency response methods.

The energy method treats the free vibration of a conservative system. The total mechanical energy is the sum of the kinetic energy T and the potential energy U. Since the total energy $(T+U)$ is constant, the equation of motion is derived from

$$\frac{d}{dt}(T+U)=0 \tag{2-3}$$

Rayleigh's method assumes that (1) the motion is sinusoidal, and (2) the maximum kinetic energy is equal to the maximum potential energy. Thus, if the mass-of a spring is not negligible, an equivalent mass m_{eq} of the spring can be calculated from its kinetic energy. Hence the natural frequency of the system is $\omega_n=\sqrt{k/(m+m_{eq})}$.

The equation of motion from Newton's second law is

$$m\ddot{x}=\sum(\text{forces})_x \tag{2-16}$$

All the static forces can be neglected if $x(t)$ is measured from the static equilibrium position of the system. The general solution by the "classical" method is

$$x=x_c+x_p \tag{2-20}$$

where $x_c(t)$ is the complementary function representing the transient motion and $x_p(t)$ the particular integral due to the excitation. The roots of the characteristic equation from Eq. (2-24) or (2-29) dictate the form of $x_c(t)$. This gives the "natural" behavior of the system. If the system is underdamped, $x_c(t)$ is sinusoidal with exponentially decreasing amplitude as shown in Eq. (2-35).

If the excitation is harmonic, $x_p(t)$ is harmonic and at the excitation frequency. The harmonic response, described by Eq. (2-38), is shown graphically in Figs. 2-8 to 2-12. Many schemes can be used for this graphical presentation.

The general solution in Eq. (2-47) shows that the arbitrary constants A and ψ occur only in $x_c(t)$. They are evaluated by applying the initial conditions to the general solution, since it is the entire solution that must satisfy the initial conditions.

In the frequency response method, vectors are used to represent the sinusoidal functions in the equation of motion. Denoting the excitation by the vector $\bar{F}e^{j\omega t}$ and substituting $j\omega$ for d/dt in the equation of motion, the amplitude X and the phase angle ϕ of the steady-state response

$x(t) = X\sin(\omega t - \phi)$ are as shown in Eqs. (2-53) and (2-54). The mechanical impedance and the sinusoidal transfer function methods are variations of this technique, although they are very important in vibration measurement.

An arbitrary excitation $F(t)$ can be approximated by a sequence of pulses as shown in Fig. 2-20(*a*). The system response due to $F(t)$ is the sum of the'responses due to the individual pulses. In other words, if the impluse response in Eq. (2-69) of a system is known, its response due to $F(t)$ can be obtained by superposition. This gives the convolution integral in Eq. (2-71).

Zero initial conditions are generally assumed when applying the convolution integral. If the initial conditions are not zero, the complementary solution in Eq. (2-73) due to the initial conditions is added directly to yield the complete solution in Eq. (2-74). In contrast, the classical method first obtains the general solution of the nonhomogeneous equation and then evaluates the constants of integration by applying the initial conditions to the general solution.

The theory discussed in this chapter is equally applicable to systems with rectilinear and rotational motions. The two types of systems are compared in Tables 2-2 and 2-3.

PROBLEMS

Assume all the systems in the figures to follow are shown in their static equilibrium positions.

2-1 Use the energy method to determine the equations of motion and the natural frequencies of the systems shown in the following figures:

(a) Figure 2-1(*b*). Assume the mass of the torsional bar k, is negligible.

(b) Figure 2-1(*d*). Assume there is no slippage between the cord and the pulley.

(c) Figure 2-1(*f*). Consider the mass of the uniform rod L.

(d) Figure P2-1(*a*). Assume there is no slippage between the roller and the surface.

(e) Figure P2-1(*b*). Assume there is no slippage between the roller and the surface. Neglect the springs k_2 and let the springs k_1 be under initial tension.

(f) Repeat part e. including springs k, and k_2. Assume all the springs are under initial compression.

(g) Figure P2-1(*c*). Assume there is no slippage between the pulley and the cord.

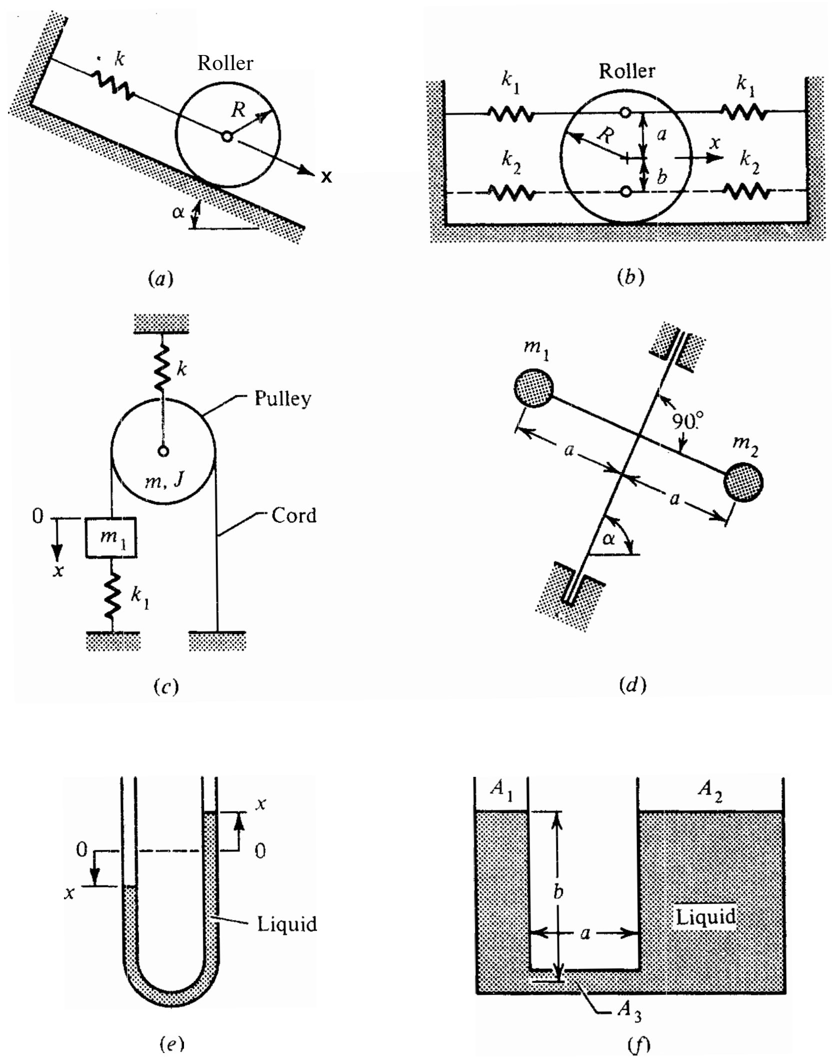


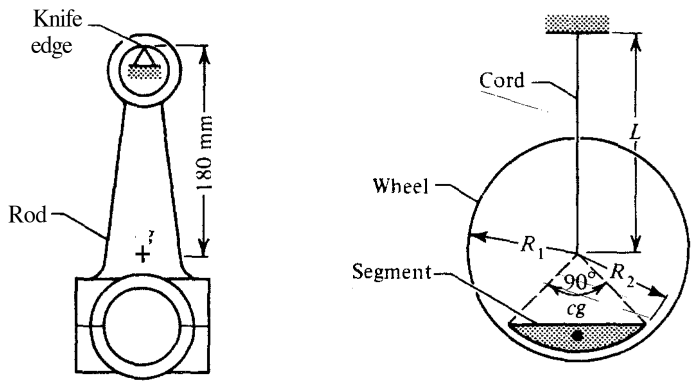
FIG. P2-1. Vibratory *systems.*

(h) Figure P2-1(*d*). Assume $m_2 > m_1$.

(i) Figure P2-1(*e*). The U tube is of uniform cross section.

(j) Figure P2-1(*f*). The cross sectional areas are as indicated.

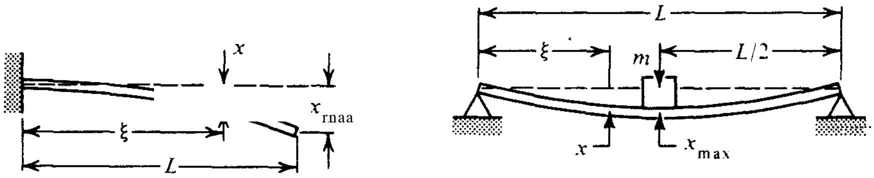
2-2 A connecting rod of 2.0 kg mass is suspended on a knife edge as shown in Fig. P2-2(*a*). If the period of oscillation is 1.2 s, find the mass moment of inertia J_{cg} of the rod about its mass center cg.



(a) Connecting rod (b) Wheel and segment

FIG. P2-2. Pendulums.

2-3 A counter weight in the form of a circular segment as shown in Fig. P2-2(*b*) is attached to a uniform wheel. The mass of the wheel is 45 kg and that of the segment 4 kg. The wheel-and-segment assembly is swung as a pendulum. If $R_1 = 250$ mm, $R_2 = 230$ mm, and $L = 500$ mm, find the period of the oscillation.



(a) Cantilever (b) Simply supported beam

FIG. P2-3. *Fundamental* frequency of beams.

2-4 A uniform cantilever beam of ρ mass/length is shown in Fig. P2-3(*a*). Assume that the beam deflection during vibration is the same as its deflection for a concentrated load at the free end, that is, $x = \frac{1}{2}x_{max}[3(\xi/L)^2 - (\xi/L)^3]$. (a) Determine the natural frequency of the beam. *(b)* Define an equivalent mass at the free end of the beam for this mode of vibration.

2-5 Repeat Prob. *2-4* if the deflection curve is assumed as $x/x_{max} = \xi/L$. What is the percentage error in the natural frequency as compared with Prob. *2-4?* Note that the assumed deflection curve does not satisfy the boundary condition at the fixed end, since the slope at the fixed end must be zero.

2-6 Repeat Prob. *2-4* if a mass m is attached to the free end of the cantilever.

2-7 A simply supported uniform beam with a mass m attached at midspan is shown in Fig. P2-3(*b*). The mass of the beam is ρ mass/length. Assume that the deflection during vibration is the same as the static deflection for a concentrated load at midspan, that is, $x = x_{max}[3(\xi/L) - 4(\xi/L)^3]$ for $0 \le \xi \le L/2$. (a) Find the fundamental frequency of the system. *(b)* What is the equivalent mass of the beam at $L/2$?

2-8 Repeat Prob. 2-7 if the deflection curve is assumed $x/x_{max} = \xi/(L/2)$ for $0 \le \xi \le L/2$.

2-9 A uniform bar of ρ mass/length with an attached rigid mass m is shown in Fig. P2-4(a). Assume the elongation of the bar is linear, that is, $x/x_{max} = \xi/L$. Find the frequency for the longitudinal vibration of the bar.

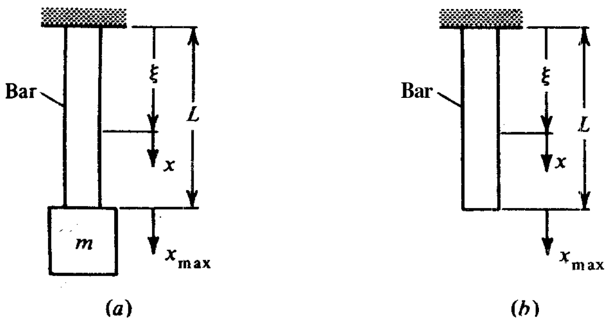


FIG. P2-4. Fundamental *frequency* of bars.

2-10 A uniform bar of ρ mass/length is shown in Fig. P2-4(b). Assume the maximum deflection of the bar is due to its own weight. Find the fundamental frequency for the longitudinal vibration.

2-11 Repeat parts (a) to (j) of Prob. 2-1, using Newton's law of motion.

2-12 Referring to Fig. 2-4, let the cylinder be of mass $m = 45$ kg, $R_1 = 120$ mm, and $R = 500$ mm. (a) Derive the equation of motion by means of Newton's second law. (b) Find the natural frequency.

2-13 Find the solutions of the homogeneous equation $\ddot{x} + 9x = 0$ for the following initial conditions:

(a) $x(0) = 1$ and $\dot{x}(0) = 0$

(b) $x(0) = 0$ and $\dot{x}(0) = 2$

2-14 Find the solutions of the homogeneous equation $\ddot{x} + 4\dot{x} + 13x = 0$ for the following initial conditions:

(a) $x(0) = 1$ and $\dot{x}(0) = 0$

(b) $x(0) = 0$ and $\dot{x}(0) = 2$

(c) $x(0) = 1$ and $\dot{x}(0) = 2$

2-15 Assuming the initial conditions $x(0) = \dot{x}(0) = 0$, find the solution of each of the following nonhomogeneous equations:

(a) $\ddot{x} + 4\dot{x} + 13x = 5e^{-t}$

(b) $\ddot{x} + 4\dot{x} + 13x = 5\sin(4t + \pi/3)$

(c) $\ddot{x} + 4\dot{x} + 13x = 5e^{-t}\sin 4t$

2-16 The equations of motion are given as (a) $m\ddot{x}+c\dot{x}+kx=F\sin(\omega t+\alpha)$, and (b) $m\ddot{x}+c\dot{x}+kx=F\cos(\omega t+\beta)$. Derive the steady-state response of each of these equations by the method of undetermined coefficients.

2-17 A machine of 20 kg mass is mounted as shown schematically in Fig. 2-7. If the total stiffness of the springs is 17 kN/m and the total damping is 300 N · s/m, find the motion $x(t)$ for the following initial conditions:

(a) $x(0)=25$ mm and $\dot{x}(0)=0$

(b) $x(0)=25$ mm and $\dot{x}(0)=300$ mm/s

(c) $x(0)=0$ and $\dot{x}(0)=300$ mm/s

2-18 Repeat Prob. 2-17 if an excitation force $80\cos 35t$ N is applied to the mass of the system.

2-19 An excitation of $20\sin(10t-30°)$ N is applied to the mass or a mass-spring system with $m=18$ kg and $k=7$ kN/m. (a) Find the motion $x(t)$ of the mass for the initial conditions $x(0)=\dot{x}(0)=0$. (b) Repeat part a if the damping in the system is $c=200$ N . s/m.

2-20 The equation of motion of the system in Fig. 2-6 is $m\ddot{x}+c\dot{x}+kx=F\sin\omega t$. Represent the forces by rotating vectors and indicate the positions of the vectors for the following conditions:

(a) m is moving downward and it is below the equilibrium 0.

(b) m is moving upward and it is below the equilibrium 0.

(c) m is moving upward and it is above the equilibrium 0.

(d) m is moving downward and it is above the equilibrium 0.

2-21 The equations of motion are given as (a) $m\ddot{x}+c\dot{x}+kx=F\sin(\omega t+\alpha)$, and (b) $m\ddot{x}+c\dot{x}+kx=F\cos(\omega t+\beta)$. Find the steady-state response for each of the equations by the method of mechanical impedance.

2-22 Find the steady-state response of the system described in Prob. 2-14 if an excitation force $F=20\cos 35t$ N is applied to the mass of the system.

2-23 A mass-spring system with damping has $m=2$ kg, $c=35$ N · s/m, $k=4$ kN/m, and an excitation $F=30\cos\omega t$ N applied to the mass. Use the mechanical impedance method to find the steady-state amplitude X and the phase angle ϕ for each of the following excitation frequencies:

(a) $\omega=6$ rad/s

(b) $\omega=\omega_n$

(c) $\omega=120$ rad/s

2-24 Derive the equations of motion for each of the systems shown in Fig. P2-5. Derive expressions for the steady-state response of the systems by the mechanical impedance method.

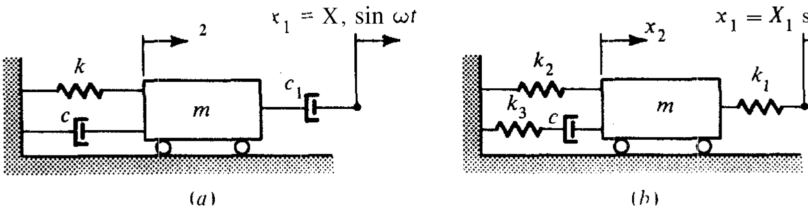


FIG. P2-5. Vibratory systems.

2-25 A constant force is applied to an underdamped mass-spring system at $t=0$. Assuming zero initial conditions, *(a)* derive the equation for the response $x(t)$, (b) find the time at which the first peak of $x(t)$ occurs, and *(c)* derive an equation relating this peak response and the damping factor ζ of the system.

2-26 Consider the equation $\tau\dot{x}+x=C$, where τ and C are **constants.** If $x(0)=0$, find the solution *(a)* by the method of undetermined coefficients, and (b) by the **convolution** integral in Eq. *(2-72)*. (c) Repeat the problem for $x(0)=x_0$.

2-27 Repeat **Prob.** *2-26* for the equation $\tau\dot{x}+x=Ct$.

2-28 Given the equation of motion of an undamped system

$$m\ddot{x}+kx=F(t) \qquad \text{or} \qquad \ddot{x}+\omega_n^2x=F(t)/m$$

derive the equation for the transient response $x(t)$ shown in Eq. *(2-74)* by *(1)* multiplying the equation above by $\sin \omega_n(t-\tau)$, and (2) integrating by parts for $0\leq\tau\leq t$, that is,

$$x(t)=x_0\cos\omega_n t+\frac{\dot{x}_0}{\omega_n}\sin\omega_n t+\frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

2-29 Given **the** equation of motion of an **underdamped system**

$$m\ddot{x}+c\dot{x}+kx=F(t) \qquad \text{or} \qquad \ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=F(t)/m$$

derive the equation for the transient response $x(t)$ shown in Eq. *(2-74)* by *(1)* multiplying the equation above by $e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)$ **and** *(2)* **integrating** by parts for $0\leq\tau\leq t$, that is,

$$x(t)=e^{-\zeta\omega_n t}\left(x_0\cos\omega_d t+\frac{\dot{x}_0+\zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right)$$

$$+\frac{1}{m\omega_d}\int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau$$

2-30 Given the system equation and the initial conditions

$$\ddot{x}+10\dot{x}+100x=60 \qquad \text{and} \qquad x(0)=1,\ \dot{x}(0)=2$$

find the transient response $x(t)$ by means of Eq. *(2-74)*. Check the answer by means of the classical method.

2-31 Assuming zero initial conditions, find the transient response $x(t)$ of a system described by the equation

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=At$$

by means of Eq. (2-74), where A = constant. Use the **classical** method to check the answer.

Computer problems:

2-32 Use the program **TRESPSUB listed** in Fig. 9-2(a) to find the transient response $x(t)$ of the system

$$m\ddot{x}+c\dot{x}+kx=F(t)$$

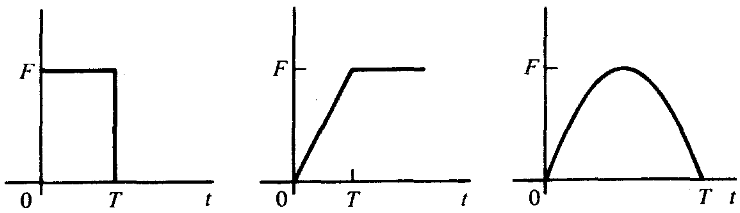
Let $F(t)$ be as shown in Fig. **P2-6(a).** Choose values for m, c, k, F, and T. Assume appropriate values for the initial conditions x_0 and $\dot{x}_0$. Select about two cycles for the duration of the run and approximately twenty data points per cycle.
Consider the problem in three parts as follows:

(a) $F(t)=0, x(0)=x_0$ and $\dot{x}(0)=\dot{x}_0$

(b) $F(t)\neq 0$, $x(0)=0$ and $\dot{x}(0)=0$

(c) $F(t)\neq 0$, $x(0)=x_0$ and $\dot{x}(0)=\dot{x}_0$

Verify from the computer print-out that $x(t)$ in part c is the sum of the parts a and b. In other words, this is to demonstrate Eq. (2-74) in which the response due to the initial conditions and the excitation can be considered separately.



(a) Rectangular pulse (b) Step input with rise time (c) A half sine pulse

FIG. **P2-6.** Excitation forces.

2-33 Repeat Prob. 2-32 for the excitation $F(t)$ shown in Fig. **P2-6(b).**

2-34 Repeat Prob. 2-32 for the excitation $F(t)$ shown in Fig. **P2-6(c).**

2-35 Select any transient excitation $F(t)$ and repeat Prob. 2-32.

2-36 It was shown in the pendulum problem in Example 1 that the equation of motion is nonlinear for large amplitudes of vibration. Consider a variation of the pendulum problem in Eq. (2-11).

$$mL^2\ddot{\theta}+c_t\dot{\theta}+mgL\sin\theta=\text{torque (t)}$$

or

$$\ddot{\theta}+2\zeta\omega_n\dot{\theta}+\omega_n^2\sin\theta=T(t)/mL^2$$

where c_t is a viscous damping factor and $T(t)$ a constant torque applied to the system. Select values for ζ, ω_n, $T(t)$, and the initial conditions $\theta(0)$ and $\dot{\theta}(0)$. Using the fourth-order Runge-Kutta method as illustrated in Fig. 9-1(a), write a **program** to implement the equation above.

2-37 (a) Repeat Prob. 2-36 but modify the program for plotting, as illustrated in Fig. 9-4(a). (b) Plot the results using **PLOTFILE listed** in Fig. 9-5(a).

3

Systems with One Degree of Freedom—Applications

3-1 INTRODUCTION

This chapter is devoted to the application of the theory developed in Chap. 2 to a large class of problems, the appearance of which may differ appreciably from that of the generalized model in Fig. 2-6. The emphasis is on problem, formulation and the generalization of each type of system. The approach is to reduce the equation of the system to the form of a one-degree-of-freedom system shown in Eq. (2-18), that is,

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F_{eq}(t) \tag{3-1}$$

where m_{eq}, c_{eq}, k_{eq}, and $F_{eq}(t)$ are the equivalent mass, damping, coefficient, spring constant, and excitation force, respectively.* Once the equation is developed, the interpretation follows the general theory discussed in the last chapter.

The equivalent quantities in Eq. (3-1) may be self-evident for simple problems. For example, the equivalent spring force $k_{eq}x$ is that which tends to restore the mass to its equilibrium position. The restoring force can be due to a spring, gravitation, the buoyancy of a liquid., a centrifugal field, or their combinations. Alternatively, from energy considerations, k_{eq} is a quantity in the total potential energy $\frac{1}{2}k_{eq}x^2$ of the system due to a displacement in the x direction. Similarly, m_{eq} is a quantity in the total kinetic energy $\frac{1}{2}m_{eq}\dot{x}^2$ of the system due to $\dot{x}$. The c_{eq} accounts for all the energy dissipation associated with x. $F_{eq}(t)$ could be due to a force

*For convenience of writing, the subscript (eq) is omitted from subsequent equations unless ambiguity arises.

and/or a motion applied to the system or an unbalance in the machine. The product of F_{eq} and the displacement x has the unit of work."

The generalized model shown in Fig. 2-6 consists of four elements, namely, the mass, the damper, the spring, and the excitation. The systems considered in this chapter are grouped according to the elements involved. If a system does not possess one of these elements, such as a damper, it is simply omitted from the equation of motion in Eq. (3-1). We shall begin with the simple mass-spring system.

3-2 UNDAMPED FREE VIBRATION

The simplest vibratory system is one that consists of a mass and a spring element. If a system is lightly damped, it can be approximated by a simple spring-mass system. Neglecting the damper and the excitation, Eq. (3-1) becomes

$$m\ddot{x} + kx = 0 \tag{3-2}$$

From Eq. (2-8), the solution of Eq. (3-2) is

$$x = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

where A_1 and A_2 are constants. Substituting the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ gives

$$x = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \tag{3-3}$$

$$x = A \sin(\omega_n t + \psi) \tag{3-4}$$

where

$$A = \sqrt{x_0^2 + (\dot{x}_0/\omega_n)^2} \quad \text{and} \quad \psi = \tan^{-1} \frac{x_0}{\dot{x}_0/\omega_n} \tag{3-5}$$

Example 1. *Equivalent* Mass

A machine component at its static equilibrium position is represented by a uniform bar of mass m and length L and a spring k in Fig. 3-1(a). Derive the equivalent system shown in Fig. 3-1(b).

Solution:

The equivalent mass m_{eq} is obtained by considering the kinetic energy T of the system as illustrated in Example 3, Chap. 2. Assuming the spring is of

* Although the concept of equivalent quantities may not be fully utilized in this chapter, they are introduced early in the text because (1) the one-degree-of-freedom system is basic in vibration, and (2) the concept of equivalent or generalized quantities is essential for more advanced studies in later chapters.

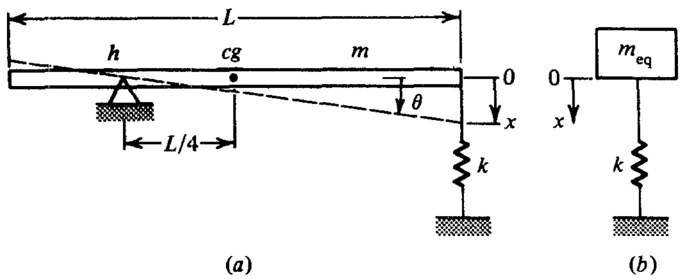


FIG. 3-1. *Equivalent mass* m_{eq}.

negligilble mass, we get

$$T = \tfrac{1}{2}J_h\dot{\theta}^2 - \tfrac{1}{2}m_{eq}\dot{x}^2$$

where

$$J_h = J_{cg} + m\left(\frac{L}{4}\right)^2 = mL^2(\tfrac{1}{12}+\tfrac{1}{16}) = \tfrac{7}{48}mL^2$$

Substituting $\dot{x} \simeq (3/4)L\dot{\theta}$ in the kinetic energy equation yields

$$m_{eq} = \tfrac{7}{27}m$$

Example 2. Equivalent Mass Moment of Inertia

A pinion-and-gear assembly is shown in Fig. 3-2. It is often convenient to refer the mass moment of inertia J of the assembly to a common shaft. Find the J_{eq} of the assembly referring to the motor shaft.

Solution:

Let N_1 be the number of teeth on the pinion and N_2 that of the gear. The gear ratio is $n = N_1/N_2$. Let θ_1 and θ_2 be the angular rotations of the pinion J_1 and the gear J_2, respectively. The kinetic energy T of the assembly referred to the motor shaft is

$$T = \tfrac{1}{2}J_1\dot{\theta}_1^2 + \tfrac{1}{2}J_2\dot{\theta}_2^2 \equiv \tfrac{1}{2}J_{eq}\dot{\theta}_1^2$$

Substituting $\dot{\theta}_2 = n\dot{\theta}_1$ in T, we get

$$J_{eq} = J_1 + n^2J_2$$

Hence the equivalent mass moment of inertia of J_2 referring to the motor shaft 1 is n^2J_2.

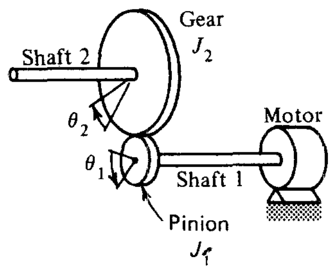
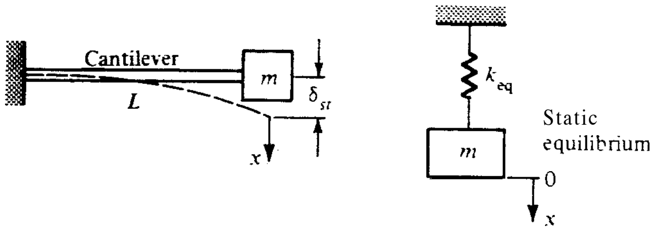


FIG. 3-2. *Equivalent mass moment of inertia* J_{eq}.



(a) Vibratory system (*b*) Equivalent system

FIG. 3-3. *Equivalent spring.*

Similarly, equivalent springs can be calculated. Let k_{t2} be the torsional **stiffness** of shaft 2. It can be shown by equating potential energies that the equivalent stiffness of shaft 2 referring to shaft 1 is n^2k_{t2}. The equivalent spring in a system can assume various forms. We shall illustrate the equivalent spring with the examples to follow.

Example 3. ***Equivalent Spring***

Figure **3-3** shows that the static deflection δ_{st} of a cantilever beam is due to the mass m attached to its free end. Find the natural frequency of the system.

Solution:

The equivalent system is as shown in Fig. **3-3(*b*)** if **(1)** the cantilever is of negligible mass and **(2)** m **is** small in size compared with L. The static **deflection** δ_{st} due to the **concentrated** force mg at the free end of a beam of length L is

$$\delta_{st} = \frac{mgL^3}{3EI}$$

where EI is the flexural **stiffness** of the beam. The equivalent spring **constant** k_{eq} is defined as force per unit deflection.

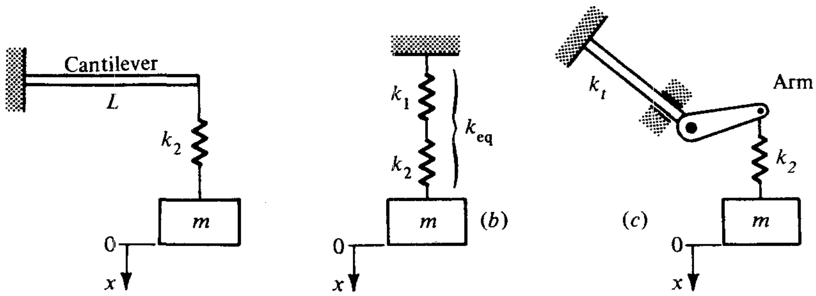
$$k_{eq} = \frac{mg}{\delta_{st}} = \frac{3EI}{L^3}$$

From the equivalent system, the natural frequency is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}} = \frac{1}{2\pi}\sqrt{\frac{3EI}{mL^3}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

Example 4. ***Springs in Series***

Springs are said to be in ***series*** when the deformation of the equivalent spring k_{eq} is the sum of their deformations. Assume the cantilever in Fig. 3-4(*a*) is of negligible mass. Show that the cantilever and the spring k_2 are in series.



(a) Vibratory system (b) and (c) Equivalent systems

FIG. *3-4. Springs in series.*

Solution:

The cantilever can be replaced by an equivalent spring k_1 of spring constant $3EI/L^3$ as in Example *3*. The equivalent system is as shown in Fig. 3-4(*b*). *A* unit static force applied at m in the **x** direction will **cause** the **spring** k_1 and k_2 to elongate by $1/k_1$ and $1/k_2$, respectively. The corresponding elongation of the equivalent spring is $1/k_{eq}$. Thus,*

$$\frac{1}{k_{eq}}=\frac{1}{k_1}+\frac{1}{k_2} \tag{3-6}$$

or

$$k_{eq}=\frac{3EIk_2}{3EI+k_2L^3}$$

The system in Fig. 3-4(*c*) consists of a torsional shaft **with** an extended arm and a spring k_2 in series. If the mass of the shaft and its arm are negligible, this system reduces to that of Fig. 3-4(*b*).

Example 5. Springs in Parallel

Springs are said to be ***in parallel*** when *(1)* the equivalent spring force is the sum of the forces of the individual springs, and (2) the springs have the same deformation. A disk *J* is connected to two shafts shown in Fig. 3-5(*a*).

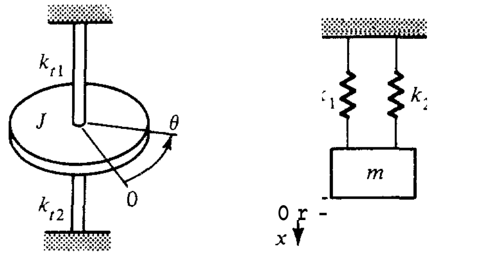
(a) Show that the shafts are equivalent to springs in parallel

(b) Determine the **natural** frequency of the system for torsional vibrations.

* Students often wonder why the equation for springs in series is like electrical impedance *Z* in parallel, where $Z=V/I$, V = voltage, and I = current. Referring to the discussion on analogy in Sec. 2-6 and using the ***current-force analogy,*** we define the impedance of a spring $Z_k=\delta/F=1/k$, where δ = deformation, F = force, and k = spring constant. It can be shown in a mechanical "circuit" that the impedance of the springs in Fig. **3-4** are in series. Thus,

$$Z_{k_{eq}}=Z_{k_1}+Z_{k_2} \quad \text{or} \quad \frac{1}{k_{eq}}=\frac{1}{k_1}+\frac{1}{k_2}$$

Equation **(3-7)** for springs in parallel can be explained in the same manner.



(a) Vibratory system (b) Equivalent system

Fig. 3-5. Springs in parallel.

Solution:

(a) Since both shafts tend to restore J to its equilibrium position, the equivalent system is as shown in Fig. 3-5(b). The restoring torque of a circular shaft is

$$T=\frac{\pi d^4 G}{32L}\,\theta=k_t\theta$$

where G is the shear modulus and d and L are the diameter and length of the shaft, respectively. If the disk **J** is rotated by an angle θ, the restoring torque T is the sum of the restoring torques of the individual shafts.

$$T=k_{eq}\theta=(k_{t1}+k_{t2})\theta$$

or

$$k_{eq}=k_{t1}+k_{t2} \tag{3-7}$$

$$k_{eq}=\frac{\pi}{32}\left(\frac{d_1^4G_1}{L_1}+\frac{d_2^4G_2}{L_2}\right)$$

(b) From the equivalent system, the natural frequency is

$$f_n=\frac{1}{2\pi}\sqrt{\frac{k_{eq}}{J}}=\frac{1}{2\pi}\sqrt{\frac{\pi}{32}\left(\frac{d_1^4G_1}{L_1}+\frac{d_2^4G_2}{L_2}\right)\frac{1}{J}}$$

Example 6. Effect of Orientation

The equivalent spring force could be due to a combination of the spring and gravitational forces. Determine the equation of motion of the systems shown in Fig. 3-6.

Solution:

Owing to the difference in the orientations of the systems, the restoring torque due to gravitation on the mass is different for the three systems. Assuming small oscillations and taking moments about 0, the equations of motion are

(a) $J_0\ddot{\theta}=\sum$ (torque),

$$mL^2\ddot{\theta}=-mgL\sin\theta-(ka\sin\theta)(a\cos\theta)$$

$$mL^2\ddot{\theta}+(mgL+ka^2)\theta=0$$

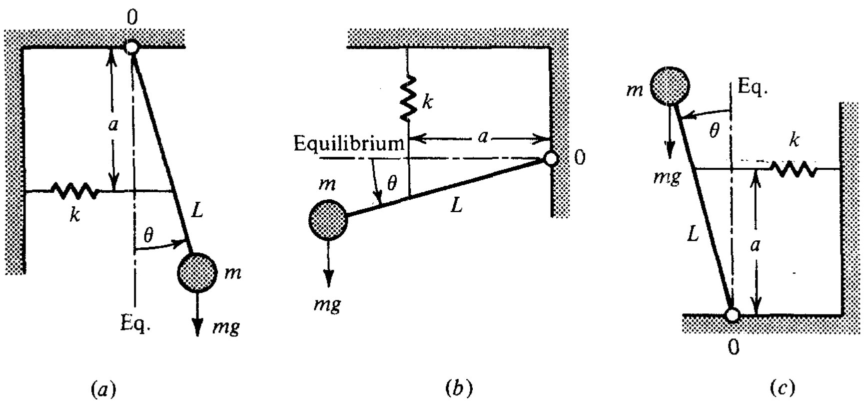


FIG. 3-6. Effect of orientation.

(b) $mL^2\ddot{\theta} = -(ka \sin\theta)(a\cos\theta)$

$mL^2\ddot{\theta} + ka^2\theta = 0$

(c) $mL^2\ddot{\theta} = mgL\sin\theta - (ka\sin\theta)(a\cos 8)$

$mL^2\ddot{\theta} + (ka^2 - mgL)\theta = 0$

Note that the quantity mg does not appear in all the equations in the examples above. Furthermore, the pendulum could be at the slant; that is, its static equilibrium positions need not be vertical or horizontal. These considerations are left as home problems.

The next two examples are variations of the pendulum problem in which the equivalent spring force is due to gravitation and a centrifugal field.

Example 7

The equation of motion of the system in Fig.. 2-4 was derived by the energy method in Example 2, Chap. 2. Derive the equation of motion by Newton's law of motion.

Solution:

Since 8, is the rotation of the cylinder relative to the curved surface, the absolute rotation of the cylinder is $(\theta_1 - \theta)$. Using the relation $R\theta = R_1\theta_1$ and taking moments about the instantaneous center of rotation b, the equation of motion is'

$$J_b(\ddot{\theta}_1 - \ddot{\theta}) = -mgR_1 \sin\theta$$

$$J_b(R/R_1 - 1)\ddot{\theta} + mgR_1 \sin 8 = 0$$

where

$$J_b = (J_0 + mR_1^2) = \left(\frac{1}{2} mR_1^2 + mR_1^2\right) = \frac{3}{2} mR_1^2$$

Assuming $\sin \theta \simeq \theta$ gives

$$\left(\frac{3}{2} mR_1^2\right)\left(\frac{R}{R_1}-1\right)\ddot{\theta}+mgR_1\theta=0$$

$$\ddot{\theta}+\frac{2g}{3(R-R_1)}\theta=0$$

Example 8. Effect of Centrifugal Field

A helicopter blade and rotor assembly is shown in Fig. 3-7(*a*). Make the necessary assumptions to simplify the problem and deduce the equation of motion for the flapping motion of the blade.

Solution:

Assume (1) the blade of mass m is a uniform bar hinged at 0, (2) the rotor angular velocity Ω is constant, and (3) the gravitational field is negligible compared with the centrifugal tield. Each element $d\xi$ of the blade is subjected to a centrifugal force $\Omega^2 R_1 \rho \, d\xi$, where p is the mass/length of the blade. The corresponding moment about 0 is $(\Omega^2 R_1 \rho \, d\xi)(\xi \sin \theta)$. Since $R_1 = R + \xi \cos \theta$, the total moment is

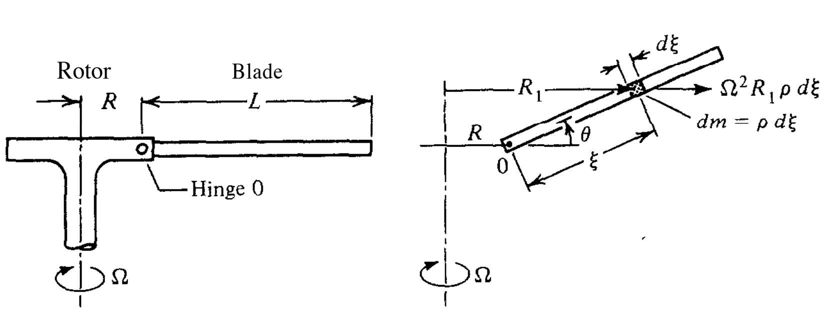
$$\int_0^L \rho\Omega^2(\sin\theta)(R+\xi\cos\theta)\xi\,d\xi=\rho\Omega^2(\sin\theta)\left(\frac{RL^2}{2}+\frac{L^3}{3}\cos\theta\right)$$

$$= m\Omega^2(RL/2+L^2/3)\theta$$

where $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$, and $m=\rho L$. The mass moment of inertia of the blade about 0 is $J_0 = mL^2/3$. Taking moments about 0 gives

$$\frac{mL^2}{3}\ddot{\theta}+m\Omega^2\left(\frac{RL}{2}+\frac{L^2}{3}\right)\theta=0$$

$$\ddot{\theta}+\Omega^2(1+3R/2L)\theta=0$$



(a) Rotor and blade (*b*) Force analysis

FIG. 3-7. *Helicopter rotor and blade.*

3-3 DAMPED-FREE VIBRATION

To a greater or lesser degree, all physical systems possess damping. For free vibration with damping, Eq. (3-1) becomes

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{3-8}$$

If the system is underdamped, from Eq. (2-34) and for the initial conditions x_0 and $\dot{x}_0$ the solution of Eq. (3-8) is

$$x = e^{-\zeta\omega_n t}\left(x_0 \cos \omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \sin \omega_d t\right)$$
$$x = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \psi) \tag{3-9}$$

where

$$A = \sqrt{(x_0\omega_d)^2 + (\dot{x}_0 + \zeta\omega_n x_0)^2}/\omega_d \tag{3-10}$$

$$\psi = \tan^{-1} \frac{x_0\omega_d}{\dot{x}_0 + \zeta\omega_n x_0} \tag{3-11}$$

and ζ, ω_n are defined in Eq. (2-27) and $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$.

Example *9*

A component of a machine is represented sch'ematically in Fig. 3-8. Derive its equation of motion.

Solution:

Assuming small oscillations and taking moments about 0, the equation of motion is

$$J_0\ddot{\theta} = \sum (\text{torque})_0$$
$$[m_1L_1^2 + m_2L_2^2 + m_3(L_3 + L_4)^2]\ddot{\theta} = m_2gL_2\theta - cL_3^2\dot{\theta} - k(L_3 + L_4)^2\theta$$
$$[m_1L_1^2 + m_2L_2^2 + m_3(L_3 + L_4)^2]\ddot{\theta} + cL_3^2\dot{\theta} + [k(L_3 + L_4)^2 - m_2gL_2]\theta = 0$$

which is of the same form as Eq. (3-8).

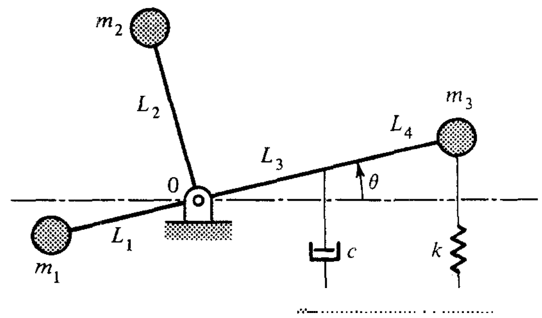


FIG. 3-8. *One-degree-of-freedom system with damping.*

Example 10. *Logarithmic Decrement*

A mass-spring system with viscous damping is shown in Fig. 2-7. The mass m is displaced by an amount x_0 from its static equilibrium position and then released with zero initial velocity. Determine the ratio of any two consecutive amplitudes.

Solution:

From Eq. (3-9), the maximum amplitude occurs when the product $Ae^{-\zeta\omega_n t}$ and $\sin(\omega_d t+\psi)$ is a maximum. Rewriting the equation with $(\omega_d t)$ as the independent variable and equating $dx/d(\omega_d t)=0$ for maximum, we have

$$x = Ae^{-\zeta\omega_d t/\sqrt{1-\zeta^2}}\sin(\omega_d t+\psi)$$

$$\frac{dx}{d\omega_d t} = Ae^{-\zeta\omega_d t/\sqrt{1-\zeta^2}}\left[-\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\psi)+\cos(\omega_d t+\psi)\right] = 0$$

Hence the maximum amplitude occurs at

$$\tan(\omega_d t+\psi)=\sqrt{1-\zeta^2}/\zeta$$

Let the motion $x(t)$ be as illustrated in Fig. 3-9 and let $\omega_d t_1$ and $\omega_d t_2$ correspond to the maxima x_1 and x_2. The last equation indicates that $\tan(\omega_d t_1+\psi)=\tan(\omega_d t_2+\psi)$. Hence $(t_2-t_1)=2\pi/\omega_d$ is a period and $\sin(\omega_d t_1+\psi)=\sin(\omega_d t_2+\psi)$. The consecutive amplitude ratio is

$$\frac{x_1}{x_2}=\frac{Ae^{-\zeta\omega_n t_1}}{Ae^{-\zeta\omega_n t_2}}=e^{\zeta\omega_n(t_2-t_1)}$$

$$\frac{x_1}{x_2}=e^{\zeta\omega_n(2\pi/\omega_d)}=e^{2\pi\zeta/\sqrt{1-\zeta^2}}$$

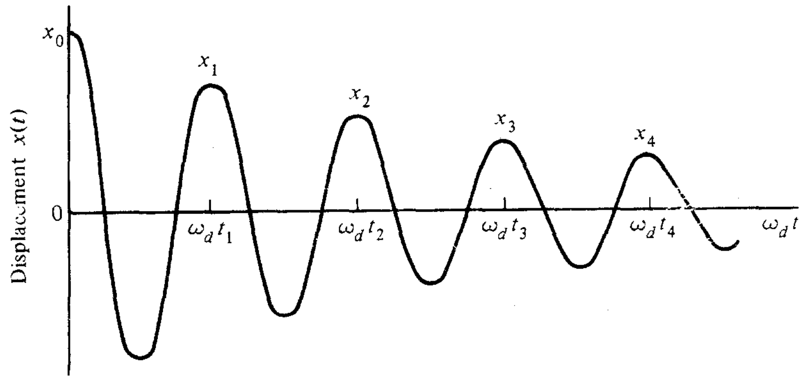


FIG. 3-9. Free vibration with damping: initial conditions $x(0)=x_0$ and $\dot{x}(0)=0$.

The natural logarithm of this ratio is called the *logarithmic decrement* 6, that is, $\ln(x_1/x_2) = \delta$. Hence

$$\delta = 2\pi\zeta/\sqrt{1-\zeta^2} \tag{3-12}$$

or

$$\delta \simeq 2\pi\zeta \quad \text{for} \quad \zeta \ll 1 \tag{3-13}$$

The logarithmic decrement is a measure of the damping factor ζ and it gives a convenient method to measure the damping in a system.

It was pointed out in Chap. 2 that the rate of decay of the vibrations is a property of the system. Hence the logarithmic decrement must be independent of initial conditions. Furthermore, any two points on the curve in Fig. **3-9** one period apart may serve to evaluate the logarithmic decrement. The use of consecutive amplitudes, however, is a convenience.

Example 11

The following data are given for a system with viscous damping: mass $m = 4$ kg (9 lb,), spring constant $k = 5$ kN/m (28 lb_f/in.), and the amplitude decreases to 0.25 of the initial value after five consecutive cycles. Find the damping coefficient of the damper.

Solution:

The amplitude ratio of any two consecutive amplitudes is

$$\frac{x_0}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = e^{\delta}$$

Hence

$$\frac{x_0}{x_5} = \frac{x_0}{x_1} \cdot \frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdot \frac{x_3}{x_4} \cdot \frac{x_4}{x_5} = e^{5\delta} = \frac{1}{0.25}$$

$$\delta = \frac{1}{5} \ln 4 = 0.277 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

The damping factor ζ and the damping coefficient c are

$$\zeta = 0.044$$

$$c = 2\zeta\sqrt{km} = 2(0.044)\sqrt{5000(4)} = 12.5 \text{ s . N/m}$$

Following the method in the example above, the nunnber of cycles n required to reduce the amplitude by a factor of N is given by the expression

$$\frac{x_0}{x_n} = N = e^{n\delta} \quad \text{or} \quad \delta = \frac{1}{n} \ln N \tag{3-14}$$

Assuming $\delta \simeq 2\pi\zeta$, Eq. (3-14) is plotted in Fig. **3-10.** Note that it takes

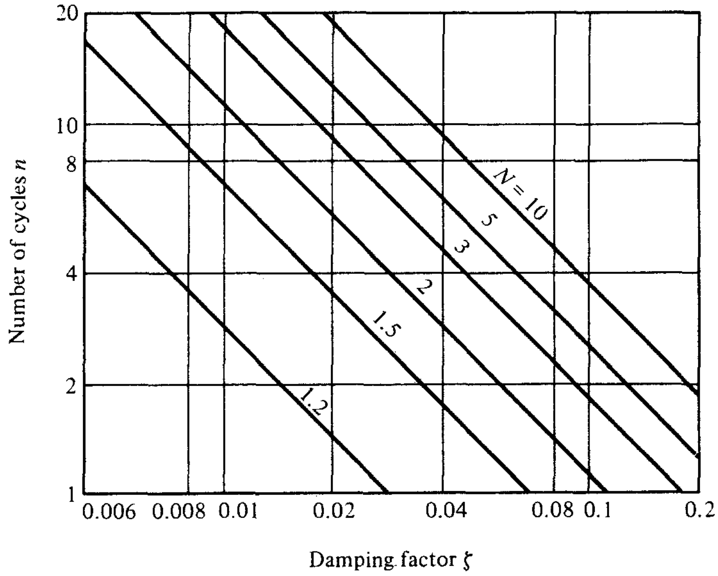


FIG. 3-10. *Number of cycles n to reduce the amplitude by a factor of N for small values of damping.*

less than four cycles to attenuate the amplitude by a factor of 10 when $\zeta = 0.1$. This implies that it does not take many cycles for the transient motion to die out even for lightly damped systems.

3-4 UNDAMPED FORCED VIBRATION—HARMONIC EXCITATION

The usual interest in the study of forced vibration with harmonic excitation is the steady-state response of the system. As discussed in the last section, the transient motion will soon die out, even for lightly damped systems. Except near resonance, the steady-state response can be approximated by that of an undamped system.

Neglecting the damping term in Eq. (3-1), the equation of motion for a system with harmonic excitation is

$$m\ddot{x} + kx = F \sin \omega t \tag{3-15}$$

If the transient motion is assumed to have died out, the steady-state response from Eq. (2-42) is

$$x = \frac{F/k}{1 - r^2} \sin \omega t \tag{3-16}$$

where $r = \omega/\omega_n$ and $\omega_n^2 = k/m$.

The equation above is plotted in Figs. 2-8 to 2-12. The corresponding curves are when $\zeta = 0$. Resonance occurs when the frequency ratio $r = 1$.

The amplification factor R is infinite at resonance and the amplitude of the displacement also becomes infinite.

Alternatively, the behavior at resonance can be deduced from the particular integral of Eq. (3-15). Dividing the equation by m and substituting ω_n for ω in the excitation term, we obtain

$$\ddot{x} + \omega_n^2 x = \frac{F}{m} \sin \omega_n t \tag{3-17}$$

From Example 2, App. D, the particular integral is of the form

$$x = A_1 t \sin \omega_n t + A_2 t \cos \omega_n t$$

where A, and A, are undetermined coefficients. Substituting this into Eq. (3-17) and solving for the coefficients, we get

$$x = -\frac{F}{2\sqrt{km}} t \cos \omega_n t \tag{3-18}$$

Thus, the amplitude increases proportionately with time and would theoretically become infinite.

Equation (3-18) indicates that it takes time for the amplitude to build up at resonance. Hence, if the resonance is passed through rapidly, it is possible to bring up the speed of a machine, such as a turbine, to beyond resonance or its *critical* speed. At frequencies considerably above resonance, Fig. 2-8 shows that the magnification factor is less than unity. It may be advantageous to operate the machine in this speed range. It should be cautioned that during the shut-down the machine would again pass through the critical speed and excessive vibration might be encountered.

Example 12. Determination of Natural Frequency

The control tab of an airplane elevator is shown schematically in Fig. 3-11. The mass moment of inertia J_0 of the control tab about the hinge point 0 is

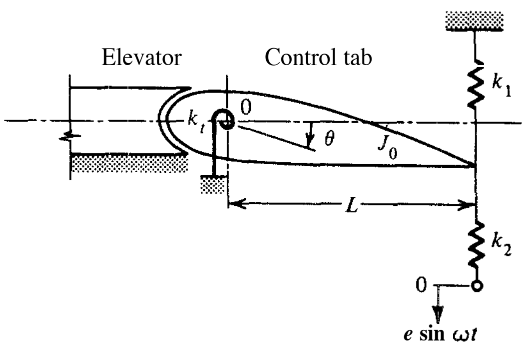


FIG. 3-11. *Determination of natural frequency.*

known, but the torsional spring constant k_t due to control linkage is difficult to evaluate. To determine the natural frequency experimentally, the elevator is rigidly mounted and the tab is excited as illustrated. The excitation frequency is varied until resonance occurs. If the resonance frequency is o,, find the natural frequency $\omega_n = \sqrt{k_t/J_0}$ of the control tab.

Solution:

Taking moments about the hinge point 0, the equation of motion of the test system is

$$J_0\ddot{\theta} = -k_t\theta - k_1L^2\theta - k_2(L\theta - e\sin\omega t)L$$

where *(LO – e* sin *wt)* is the deformation of the spring k_2. Rearranging, this equation becomes

$$J_0\ddot{\theta} + [k_t + (k_1 + k_,)L^2]8 = k_2eL \sin o t$$

At resonance,

$$\omega^2 = \omega_r^2 = \frac{k_t + (k_1 f k_2)L^2}{J_0} = \omega_n^2 + \frac{(k_1 + k_2)L^2}{J_0}$$

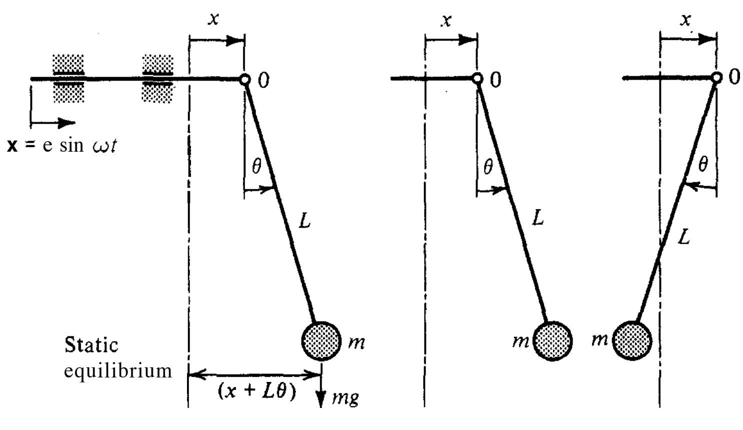
Hence

$$f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi}\sqrt{\omega_r^2 - \frac{(k_1 + k_2)L^2}{J_0}}\ \text{Hz}$$

The next three examples illustrate an application of the simple pendulum in a dynamic absorber for vibration control.

Example 13

The simple pendulum in Fig. 3-12 is hinged at the point 0. The hinge point 0 is given a horizontal motion $x(t) = e\sin\omega t$. Find (a) the angular displacement *8* of the pendulum for frequency ratios $r \lessgtr 1$, and (b) the force required to move the hinge point.



(a) Vibratory system (*b*) r < 1 (*c*) *r* > 1

FIG. 3-12. *Pendulum* excited *at support.*

Solution:

(a) Assuming small oscillations, the horizontal acceleration of m in Fig. 3-12(a) is $(\ddot{x}+L\ddot{\theta})$ and the vertical acceleration is of second order. Taking moments about 0, the equation of motion is

$$m(\ddot{x}+L\ddot{\theta})L=-mgL\theta$$

$$mL^2\ddot{\theta}+mgL\theta=-mL\ddot{x}=me\omega^2L\sin\omega t \tag{3-19}$$

$$mL^2\ddot{\theta}+mgL\theta=T_{eq}\sin\omega t$$

where $T_{eq}=me\omega^2L$ is the amplitude of the equivalent torque. This equation is of the same **form** as Eq. **(3-15).** From **Eq. (3-16),** the steady-state response is

$$\theta(t)=\frac{T_{eq}/mgL}{1-r^2}\sin\omega t=\frac{(e/L)r^2}{1-r^2}\sin\omega t$$

or

$$\theta(t)=\frac{(e/L)r^2}{|1-r^2|}\sin(\omega t-\phi)$$

where $r=\omega/\omega_n$ and $\omega_n=\sqrt{g/L}$. Note that $\phi=0°$ when $r<1$ and $\phi=180°$ when $r>1$. In other words, $\theta(t)$ and $x(t)$ are in phase with one another when $r<1$ and $180°$ out of phase when $r>1$. These phase relations are illustrated in Figs. 3-12(b) and (c).

(b) For dynamic equilibrium of the pendulum, the horizontal force at the hinge point 0 is equal to the horizontal component of the inertia force of the pendulum, that is,

$$F_x(t)=-m(\ddot{x}+L\ddot{\theta})=mg\theta=\frac{me\omega^2}{1-r^2}\sin\omega t$$

where $x(t)$ is positive if the motion is to the right of the static equilibrium position. The equation shows that near resonance when $r\simeq 1$ a large force $F_x(t)$ could associate with a small amplitude e at the hinge point 0.

Example 14. ***Centrifugal** Pendulum Dynamic Absorber*

A simple pendulum in a centrifugal field can be used to nullify the torsional disturbing moment on a rotating machine member, such as a crank shaft. **A** rotating disk with a pendulum hinged at B is shown in **Fig. 3-13.** The disk has an average speed ω and a superimposed oscillation $\gamma=\Gamma\sin n\omega t$, where n is the number of disturbing cycles per revolution of the disk. (**a**) Derive the equation of motion of the system. (b) Find the amplitude ratio $\Gamma:\Theta$. (**c**) Briefly discuss the application of the pendulum.

Solution:

The pendulum is under the influence of a centrifugal field when the system is in rotation. Assume the gravitation is negligible compared with the

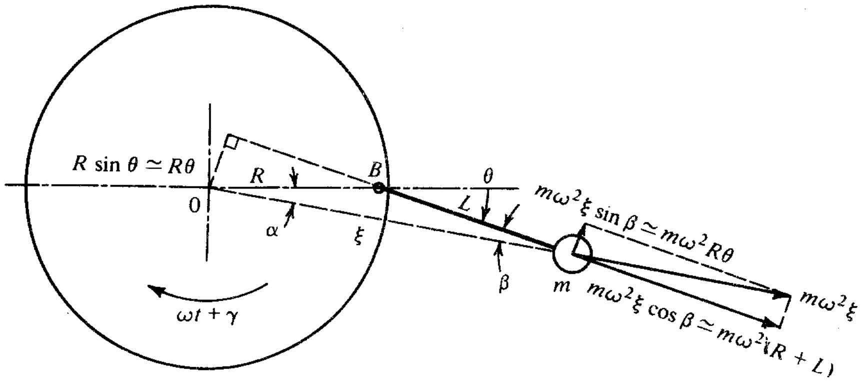


FIG. *3-13.* ***Centrifugal pendulum.***

centrifugal field at moderate and high speeds of rotation. As shown in the figure, the pendulum bob m is subjected to a centrifugal force $m\omega^2\xi$. The component of this force normal to L is $(m\omega^2\xi \sin\beta)$. From the triangle $0Bm$, we have

$$\frac{R}{\sin\beta} = \frac{5}{\sin(180-8)} = \frac{\xi}{\sin\theta} = \frac{L}{\sin a}$$

or

$$\sin\beta = \frac{R}{\xi}\sin\theta \simeq \frac{R}{\xi}\theta$$

(a) For small oscillations, the tangential acceleration of m is $[L\ddot{\theta} + (R+L)\ddot{\gamma}]$. Taking moments about B, the equation of motion of the pendulum is

$$m[L\ddot{\theta} + (R+L)\ddot{\gamma}]L + m\omega^2 RL\theta = 0$$

Since $\ddot{\gamma} = -(n\omega)^2\Gamma \sin$ *not,* the equation becomes

$$mL^2\ddot{\theta} + m\omega^2 RL\theta = m(R+L)L(n\omega)^2\Gamma \sin n\omega t$$

which is of the same form as Eq. *(3-15).*

(b) Comparing with Eq. (3-15), we have $m_{eq} = mL^2$, $k_{eq} = mw^2RL$, and $F_{eq} = m(R+L)L(n\omega)^2\Gamma$ and $\theta(t)$ is analogous to $x(t)$. The steady-state solution of the equation is

$$\theta = \Theta \sin(n\omega t - \phi)$$

The amplitude ratio Γ/Θ can be obtained by substituting the corresponding eqivalent quantities into Eq. *(3-16).* Performing the substitution and simplifying, we obtain

$$\frac{\Gamma}{\Theta} = \frac{R/L - n^2}{n^2(R+L)/L} \qquad \textbf{(3-20)}$$

(r) The equation above shows that if $n = \sqrt{R/L}$, the amplitude ratio $\Gamma/\Theta = 0$. Physically, this means that a finite value of Θ is possible for an arbitrarily small value of Γ. If the superposed oscillation $\gamma = \Gamma \sin n\omega t$ is due to a disturbing torque, the resultant oscillation $\gamma(t)$ of the machine member can be very small for a finite value of Θ. In other words, if the centrifugal pendulum is tuned such that $n = \sqrt{R/L}$, the disturbing torque can be balanced by the inertia torque of the pendulum.

The number of disturbing cycles n per revolution in a rotary machine, such as an internal combustion engine, is constant. When a centrifugal pendulum is tuned for $n = \sqrt{R/L}$, it is effective as a ***dynamic absorber*** for all speeds of operation of the machine. This is not a friction type damper, since it creates an equal and opposite torque to nullify the disturbing torque.

From Fig. 3-13, the reacting moment on the shaft is due to the tension $m\omega^2(R+L)$ of the pendulum. The moment arm $R \sin \theta \approx R\theta$. Hence the magnitude of the reacting torque T_0 is

$$T_0 = m\omega^2(R+L)R\Theta \tag{3-21}$$

Example 15

An eight cylinder, four-stroke cycle engine operates at 1,800 rpm. The fluctuating torque is absorbed by the flywheel and a dynamic absorber. Assume the disturbing torque T_0 to be balanced by the absorber is 500 N . m (4,425 lb_f-in.). The most convenient length for R as shown in Fig. 3-13 is 96 mm (3.8 in.). If the maximum amplitude of oscillation of the pendulum is 10°, find the length L and the mass m of the pendulum.

Solution:

The eight cylinder engine has four power strokes per revolution. Hence $n = 4$. The length of a properly tuned pendulum is

$$L = R/n^2 = 96/16 = 6 \text{ mm}$$

From the expression for T_0 in Example 14, the required mass m of the

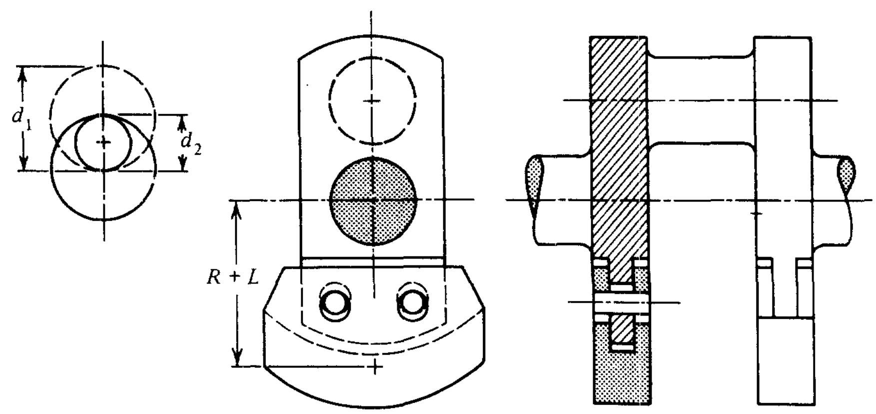


FIG. 3-14. Bifilar-type centrifugal-pendulum dynamic absorber.

pendulum is

$$m = \frac{T_0}{\omega^2(R+L)R\Theta} = \frac{500}{(60\pi)^2(0.096+0.006)(0.096)(\pi/18)}$$
$$= 8.40 \text{ kg } (18.5 \text{ lb},)$$

The length of the pendulum is small; $L = 6$ mm. This poses a design problem in providing sufficient mass with a small L. The problem is solved by using a bifilar-type centrifugal pendulum as shown in Fig. 3-14. Let half of the pendulum mass be mounted on two loosely fitted pins on each side of the crank. The diameter of the pins is d_2 and that of the holes through the mass and the crank is d_1. Thus, each point on the mass moves in an arc of a circle of radius $(d_1 - d_2)$. The length of the pendulum is $L = (d_1 - d_2)$.

3-5 DAMPED FORCED VIBRATION—HARMONIC EXCITATION

Periodic excitation generally occurs in machines under steady-state operation. Considering one of the harmonic components of the periodic excitation, the equation of motion, Eq. (3-1), becomes

$$m\ddot{x} + c\dot{x} + kx = F_{eq} \sin \omega t \tag{3-22}$$

This is identical to Eq. (2-18) except for the substitution of F_{eq} for F.

The objective is to examine the sources of F_{eq} and the manner it affects the system response $x(t)$. In other words, the aim is in problem formulation and in reducing the equation of motion to the form shown above. The solution of the equation and the interpretation of its response were fully discussed in Chap. 2. Note that the phase angle of $x(t)$ relative to F_{eq} would be identical to the previous discussions as shown in Eq. (2-54). Unless it is justifiable, the phase relation will not be discussed.

The applications are divided into six cases, which appear as distinct types of problems. The equations of motion of Case 1 to 5 are reduced to Eq. (3-22). Case 6 shows a generalization of vibration isolation. The subject is further generalized for periodic excitations in the next section.

It is advantageous to use the impedance method in Sec. 2-6 to treat the problem. Denoting $F_{eq} \sin \omega t$ by the vector $\mathbf{F}_{eq} = \bar{F}_{eq} e^{j\omega t}$ and the response $x(t) = X \sin(\omega t - \phi)$ by the vector $\mathbf{X} = \bar{X} e^{j\omega t} = (X e^{-j\phi}) e^{j\omega t}$, Eq. (3-22) becomes

$$m\ddot{x} + c\dot{x} + kx = \bar{F}_{eq} e^{j\omega t}$$

Substituting $x(t) = \bar{X} e^{j\omega t}$ gives

$$(k - \omega^2 m + j\omega c)\bar{X} e^{j\omega t} = \bar{F}_{eq} e^{j\omega t} \tag{3-23}$$

where $\bar{F}_{eq} = F_{eq} e^{j\alpha}$ is the phasor of the excitation vector $\mathbf{F}_{eq}$. The magnitude of $\mathbf{F}_{eq}$ is $F_{eq} = |\bar{F}_{eq}|$ and its phase angle is α, relative to a reference vector. If the excitation is the reference, then α is zero. Similarly, $\bar{X} = Xe^{-j\phi}$ is the phasor of the vector X. The magnitude of X is $X = |\bar{X}|$ and its phase angle is $-\phi$ relative to the reference vector. It is understood that the force and the response must be along the same axis in a physical problem. Hence it is unnecessary to denote the real or the imaginary parts of the vectors in all subsequent discussions.

Eliminating the $e^{j\omega t}$ term in Eq. (3-23) and rearranging, we get

$$\bar{X} = \frac{1}{k - \omega^2 m + j\omega c} F_{eq} \tag{3-24}$$

which is identical lo Eq. (2-52). Hence

$$X = |\bar{X}| = \frac{F_{eq}}{k} R = \frac{F_{eq}}{k} \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \tag{3-25}$$

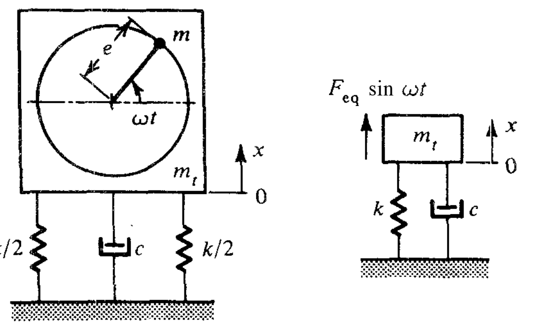
$$-\phi = \underline{/\bar{X}} = -\tan^{-1} \frac{2\zeta r}{1-r^2} \tag{3-26}$$

where $r = \omega/\omega_n$, $\omega_n^2 = k/m$, and $\zeta = \frac{1}{2}c/\sqrt{km}$.

Case 1. Rotating **and** *Reciprocating* **Unbalance**

A turbine, an electric motor, or any device with a rotor as a working part is a rotating machine. Unbalance exists if the mass center of the rotor does not coincide with the axis of rotation. The unbalance me is measured in terms of an equivalent mass m with an eccentricity **e.**

A rotating machine of total mass m, with an unbalance ***me*** is shown in Fig. 3-15. The eccentric mass m rotates with the angular velocity o and its vertical displacement is ***(x + e*** sin ***ot).*** The machine is constrained to move in the vertical direction and it has one degree of freedom. The



(a) Vibratory system (*b*) Equivalent system

Fig. 3-15. Rotating unbalance.

displacement of the mass $(m_t - \text{m})$ is $x(t)$. Hence the equation of motion of the system is

$$(m_t - m)\ddot{x} + m\frac{d^2}{dt^2}(x + e \sin \omega t) + c\dot{x} + kx = 0$$

Rearranging the equation yields

$$m_t\ddot{x} + c\dot{x} + \text{kx} = \text{mew}^2 \sin \omega t = F_{\text{eq}} \sin \mathit{cut} \tag{3-27}$$

where $F_{\text{eq}} = \text{mew}^2$ is the amplitude of the excitation force. Hence the equivalent system is as shown in Fig. 3-15(*b*). The steady-state solution is given in Eqs. (3-25) and (3-26).

From Eq. (3-25), the amplitude of the harmonic response is

$$X = \frac{F_{\text{eq}}}{k} R = \frac{me\omega^2}{k} R$$

This can be expressed in a nondimensional form. Multiplying and dividing the equation by m_t, recalling $\omega_n^2 = k/m_t$, $r = \omega/\omega_n$, and simplifying, we obtain

$$\frac{m_t X}{me} = r^2 R = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \tag{3-28}$$

This is plotted in Fig. 3-16.

At low speeds, when $r \ll 1$, the force mew^2 is small and the amplitude

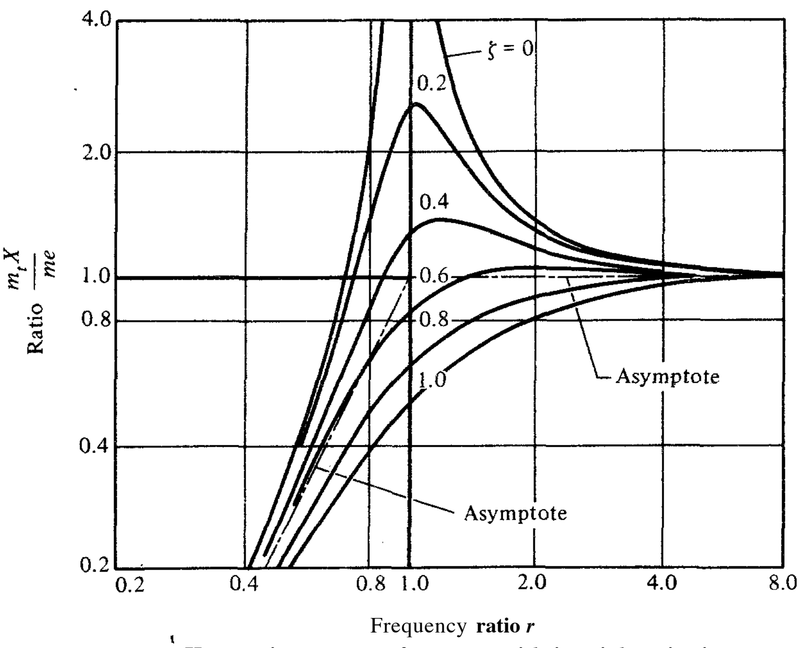


FIG. 3-16. *Harmonic response of systems with inertial excitation; system shown in Fig. 3-13.*

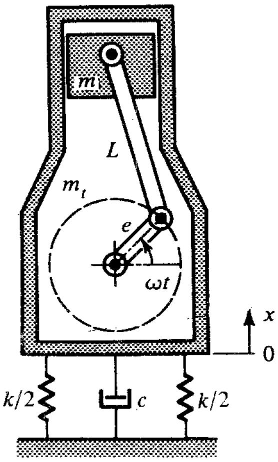


FIG. 3-17. *Reciprocating unbalance.*

of vibration X is nearly zero. At resonance, when $r = 1$ and the amplification factor $R = 1/25$, the: mass $(m_t - m)$ has an amplitude equal to $X = me/(2\zeta m_t)$. Hence the amplitude of vibration is limitedl only by the presence of **damping** in the system. Furthermore, the mass $(m_t - m)$ is 90" out of phase with the unbalance mass m. For example, **when** $(m_t - m)$ is moving **upward** and passing its static equilibrium position, the mass m is directly above its center **of** rotation. At high speeds, when $r \gg 1$, the mass $(m_t - m)$ has an amplitude $X \simeq me/m_t$. In other words, the amplitude remains **constant** independent of the frequency of excitation or the damping in the system. The phase angle is **180"**; that is, **when** $(m_t - m)$ is at its topmost position, m is directly below its center of rotation.

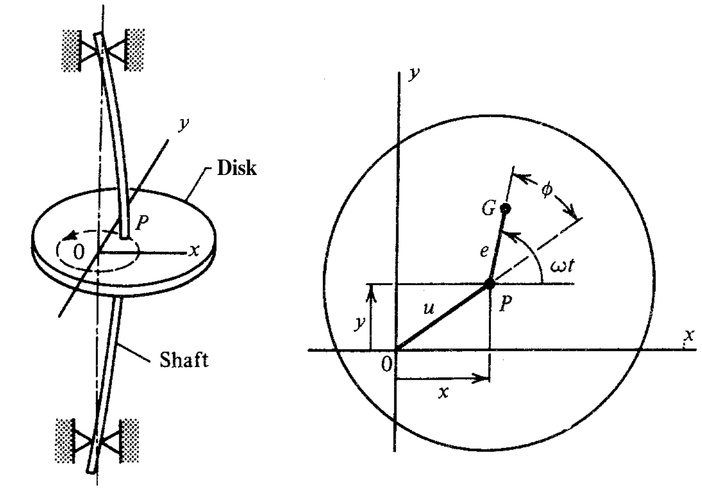
The discussion of rotating unbalance can be used to **estimate** a reciprocating unbalance. A reciprocating engine is illustrated in **Fig.** 3-17. The reciprocating mass m consists of the mass of the piston, the wrist pin, and part of the connecting rod. The exciting force is equal to the: inertia force of the reciprocating mass, which is approximately equal to $me\omega^2[\sin \omega t + (e/L)\sin 2\omega t]$,* where e is the crank radius and L the length of the connecting rod. If the ratio e/L is small, the second-harmonic term, $(e/L)\sin 2\omega t$, can be neglected.? Thus, the problem reduces to that of the rotating unbalance.

Case 2 Critical Speed of Rotating Shafts

A rotating shaft carrying an unbalance disk at its **midspan** is shown in Fig. 3-18(*a*). Critical speed occurs when the speed of rotation of the shaft

*See, for example, R. T. Hinkle, *Kinematics of Machines, 2d* ed., Prentice-Hall, Inc., Englewood Cliffs, NJ, 1960, p. 107.

† It will be shown in Case 3 for vibration isolation that if an isolator is adequate for the fundamental frequency it would be adequate for the higher harmonics.



(a) Vibratory system (b) General position of disk.

FIG. 3-18. *Critical speed of rotating shaft.*

is equal to the natural frequency of lateral (beam) vibration of the shaft. Since the shaft has distributed mass and elasticity along its length, the system has more than one degree of freedom. We assume the mass of the shaft is negligible and its lateral stiffness is k.

The top view of a general position of the rotating disk of mass m is shown in Fig. 3-18(*b*). Let G be the ***mass center*** of the disk, P the geometric center, and O the center of rotation. Assume the damping force, such as air friction opposing the shaft whirl, is proportional to the linear speed of the geometric center P and that the flexibility of the bearings is negligible compared with that of the shaft.

Resolving the forces in the x and y directions gives

$$m\frac{d^2}{dt^2}(x + e\cos\omega t) = -kx - c\dot{x}$$

$$m\frac{d^2}{dt^2}(y + e\sin\omega t) = -ky - c\dot{y}$$

$$m\ddot{x} + c\dot{x} + kx = me\omega^2\cos\omega t = F_{eq}\cos\omega t$$

$$m\ddot{y} + c\dot{y} + ky = me\omega^2\sin\omega t = F_{eq}\sin\omega t$$

Applying the impedance method illustrated in Eq. (3-23), the equations above become

$$\begin{aligned}(k - \omega^2 m + j\omega c)\bar{X} &= F_{eq}\\ (k - \omega^2 m + j\omega c)\bar{Y} &= F_{eq}e^{j\pi/2}\end{aligned} \qquad \textbf{(3-29)}$$

The phase angle $\pi/2$ in the second equation above indicates that the

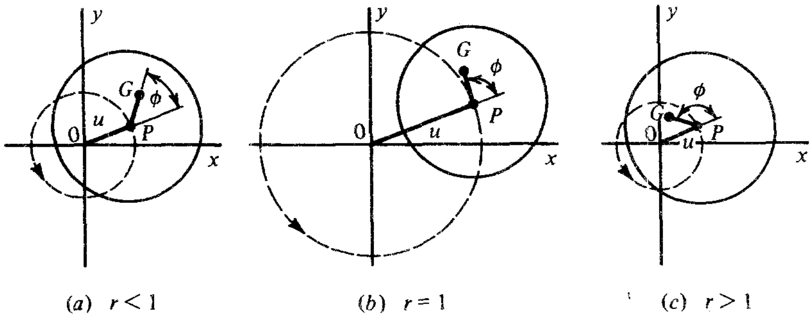


FIG. 3-19. *Phase relation of rotating shaft for* $r \lesseqgtr 1$; $r = \omega/\omega_n$.

displacements in the generic x and y directions are at 90° with one another. It is evident that the amplitudes X and Y are equal.

Since the two harmonic motions $x(t)$ and $y(t)$ are equal in magnitude, of the same frequency, and at 90° to each other, their sum is a circle. Thus, the radius u of the circle is equal to X or Y. The motion of the geometric center P of the disk in Fig. 3-18 describes a circle of radius u about the center of rotation O. From Eq. (3-25), we get.

$$u = X = Y = \frac{F_{eq}}{k} R = \frac{me\omega^2}{k} R$$

Substituting $\omega_n^2 = k/m$ and $r = \omega/\omega_n$ and simplifying we obtain

$$\frac{u}{e} = r^2 R = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \tag{3-30}$$

This is identical to Eq. (3-28) if m_t = m. This is true because the total mass is also the eccentric mass for a rotating disk. Hence the response curves in Fig. 3-16 for rotating unbalance also represents Eq. (3-30) for the whirling of rotating shafts.

The phase relation for various operating frequencies is shown in Fig. 3-19. It is interesting to note that, when the frequency ratio $r \gg 1$, the mass center G tends to coincide with the center of rotation O. This can be demonstrated readily. Assume that an unbalance rotor is rotating in a balancing machine and that a piece of chalk is moved towards the rotor until it barely touches. When the rotational speed is below critical, the chalk mark is found on the side closer to the mass center of the rotor. When the speed is above critical, the-chalk mark is on the side away from the mass center.

Example 16. *Elasticity of Bearings and Supports*

Rigid bearings were assumed in the above discussion of critical speed. Figure 3-20 shows a pulley assembly, in which the supporting brackets can be deflected more easily in the vertical direction than in the lateral

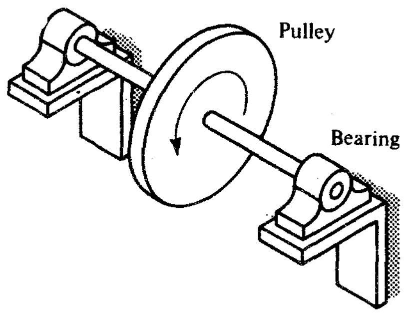
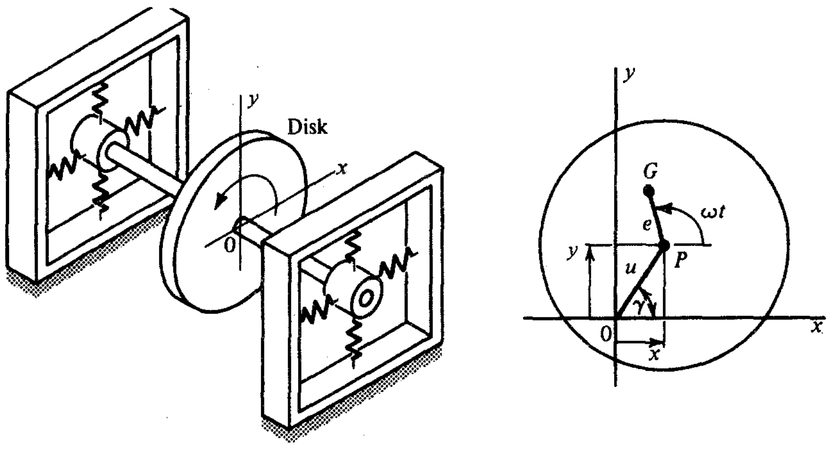


FIG. 3-20. *Pulley assembly with flexible bearing supports.*

direction. The effect of the elastic bearings is to render the system more flexible and therefore lowering the critical speed. The critical speed can be lowered by 25 percent in some installations. **(a)** Derive the equation of motion of the system and (**b**) briefly discuss the effect of the unequal elastic supports on the system performance.

Solution:

A schematic representation of the system is shown in Fig. **3-21(*a*)**. The elasticity of the bearings and supports is represented by springs mounted in rigid frames. The equivalent spring constants k_x and k_y are due to the stiffness of the shaft, the bearings, and the supports in the x and y directions. A general position of the disk is shown in Fig. **3-21(*b*)**, which may be compared with Fig. **3-18(*b*)**. P is the geometric center and G the mass center of the disk. O the center of rotation of the system corresponding to the static equilibrium position of the shaft.



(a) Schematic (b) General position of rotating disk.

FIG. 3-21. System with elastic bearing supports.

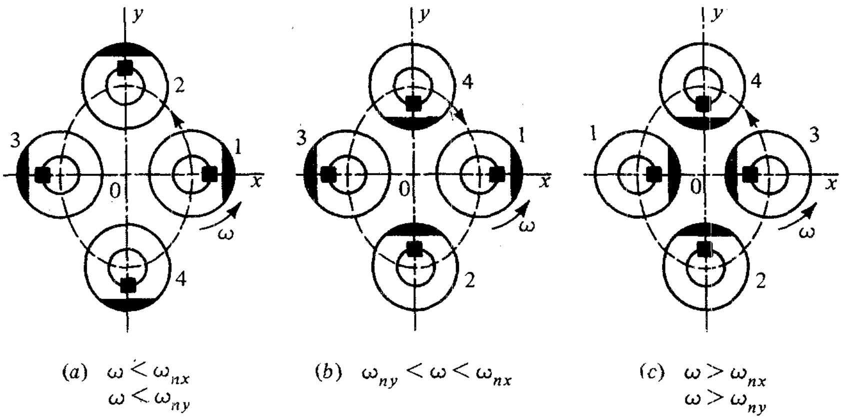


FIG. *3-22. Rotation of* **disk** *about* 0 *for various frequencies.*

(a) For simplicity, the system is assumed undamped. The equations of motion are

$$m\ddot{x} + k_x x = me\omega^2 \cos \omega t$$

$$m\ddot{y} + k_y y = me\omega^2 \sin \omega t$$

Since $k_x \neq k_y$, the equations indicate that the system has two natural frequencies and therefore two critical speeds. We define $\omega_{nx} = \sqrt{k_x/m}$, $\omega_{ny} = \sqrt{k_y/m}$, $r_x = \omega/\omega_{nx}$, $r_y = \omega/\omega_{ny}$, and $F_{eq} = mew^2$. From Eq. (3-30), the amplitude ratios are

$$\frac{X}{e} = \frac{r_x^2}{|1 - r_x^2|} \quad \text{and} \quad \frac{Y}{e} = \frac{r_y^2}{|1 - r_y^2|}$$

(b) The two harmonic motions $x(t)$ and $y(t)$ are of the same frequency and at 90° to each other. Since their amplitudes are unequal, the sum of their motions is an ellipse. Thus, the geometric center P moves in an ellipse about O as shown in Fig. 3-22. By neglecting the damping in the system, the phase angle can be either zero for below the critical speed or 180° for above the critical speed. Since there are two natural frequencies, we may consider the operations at speeds above and below the critical.

When $\omega < \omega_{nx}$ and $\omega < \omega_{ny}$, both the disk and P rotate in the same direction with the same speed as shown in Fig. 3-22(a). The heavy side of the disk and the position of the shaft key are marked for purpose of identification. Assume $\omega_{nx} > \omega_{ny}$. When $\omega_{nx} > \omega > \omega_{ny}$, the disk and P rotate in opposite directions with the same speed as shown in Fig. 3-22(b). When ω is greater than both natural frequencies, again the disk and P rotate in the same direction with the same speed as shown in Fig. 3-22(c).

It is interesting to note that when the excitation is above or below the critical speeds, there is no reversal in stresses in the shaft; that is, while the shaft is revolving, the compression side of the shaft remains in compression

and the tension side remains in tension. When the excitation is between the two critical speeds, the shaft undergoes two reversals in stress per revolution.

The balancing of machines and field balancing' are further examples of critical speed calculations. Since the subject is usually covered in the dynamics of machines, it will not be pursued here.

Case 3. Vibration Isolation and Transmissibility

Machines are often mounted on springs and dampers as shown in Fig. 3-23 to minimize the transmission of forces between the machine m and its foundation.

We shall first consider the system illustrated in Fig. 3-23(*a*). If a harmonic force is applied to m and the deflection of the foundation is negligible, the equation of motion is identical to Eq. (3-22). The force transmitted to the foundation is the sum of the spring force kx and the damping force $c\dot{x}$.

$$\text{Force transmitted} = kx + c\dot{x}$$

If the excitation is harmonic, the magnitude and the phase angle of the excitation force F_{eq} and the other forces are as illustrated in Fig. 3-24. The phase angle γ is generally is of secondary interest. Using Eq. (3-24), the force transmitted $\bar{F}_T$ is

$$\bar{F}_T = k\bar{X} + j\omega c\bar{X} = \frac{k + j\omega c}{k - \omega^2 m + j\omega c}\bar{F}_{eq} \qquad \textbf{(3-31)}$$

The ratio of the amplitude of the force transmitted F_T and the amplitude of the driving force F_{eq} is called the ***transmissibility*** TR. From the equation above, we have

$$\text{TR} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \qquad \textbf{(3-32)}$$

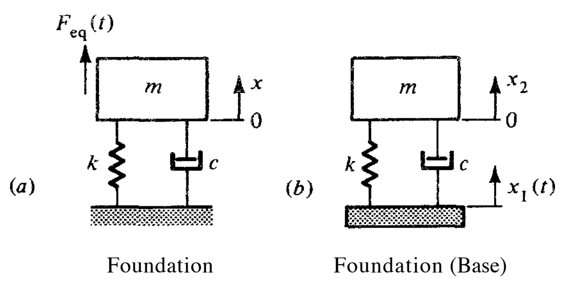


FIG. **3-23.** ***Vibration isolation.***

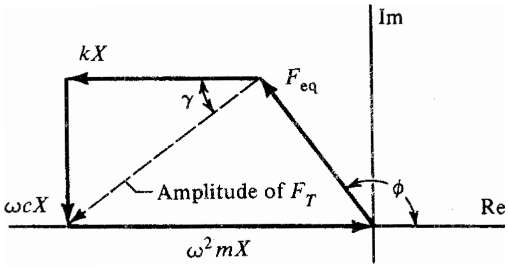


Fig. 3-24. Relation of force transmitted and other force vectors.

where $r = \omega/\omega_n$ and $c\omega/k = 2\zeta r$. The equation is plotted in Fig. 3-25. Note that all the curves in the figure cross at $r = \sqrt{2}$. Hence the transmitted force is greater than the driving force below this frequency ratio and less than the driving force when the machine is operated above this frequency ratio.

For a constant speed machine, the amplitude of the exciting force F_{eq} is constant. Hence the force transmitted is proportional to the value of the

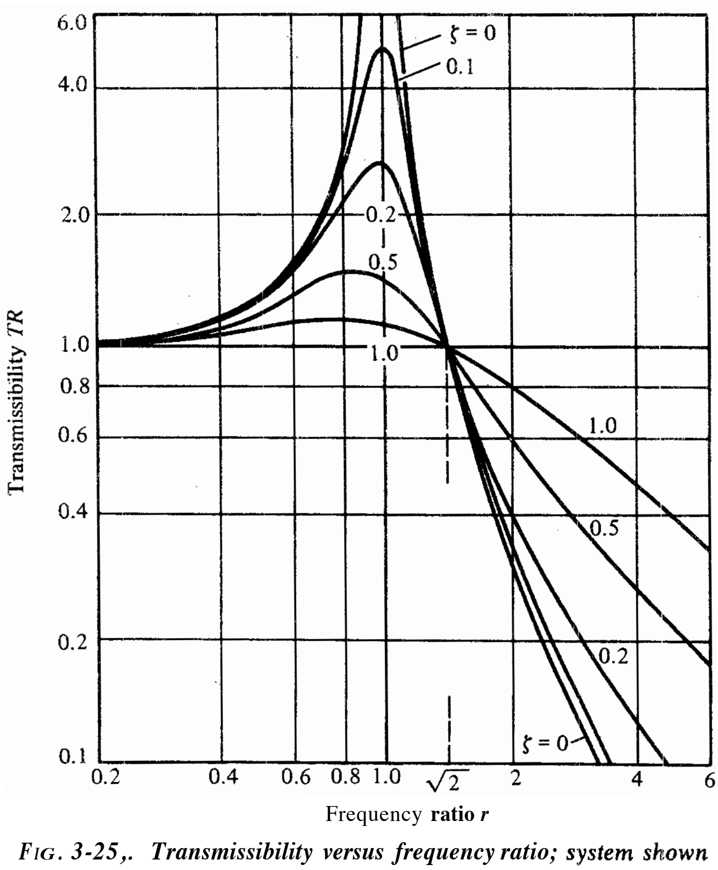


Fig. 3-25. Transmissibility versus frequency ratio; system shown in Fig. 3-23(a).

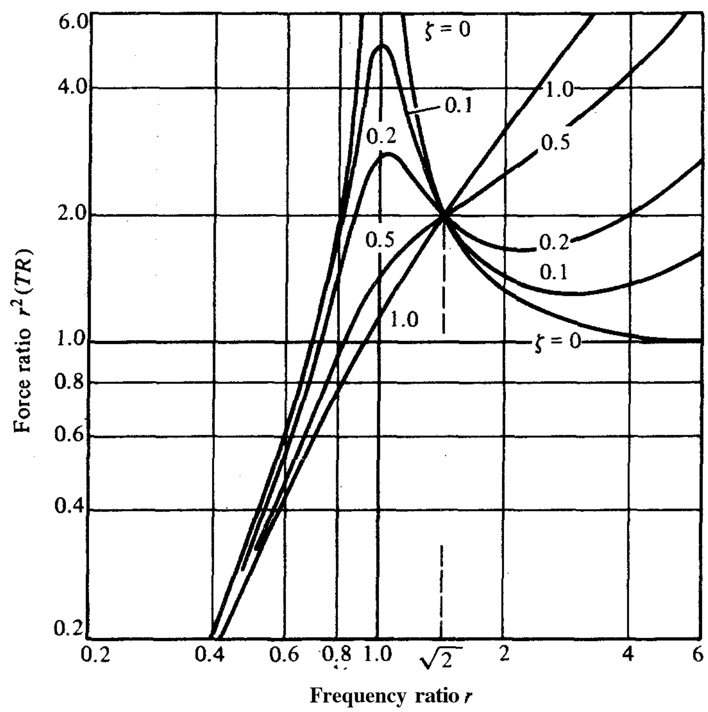


FIG. 3-26. Force ratio versus frequency ratio for inertial excitation; system shown in Fig. 3-15.

transmissibility TR. It is advantageous to operate a constant speed machine at $\omega > \sqrt{2}\omega_n$.

For a variable speed machine, the driving force F_{eq}, due to an unbalance me, is $me\omega^2$, where ω is the operating frequency. Let us define a constant force $F_n = me\omega_n^2$. Substituting $F_{eq} = me\omega^2$ into Eq. (3-31), dividing both sides of the equation by F_n, and simplifying, we obtain

$$\frac{F_T}{F_n} = \frac{r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = r^2(\text{TR}) \qquad \textbf{(3-33)}$$

where TR is as defined in Eq. (3-32). Hence the magnitude of the force transmitted can be high in spite of the low transmissibility. The equation is plotted in Fig. 3-26.

The reduction of the force transmitted in buildings is of interest. For example, the mechanical equipment of a tall office building is often located on the roof directly above the penthouse or the boardroom of the company.

The fractional reduction of the force transmitted is

$$\text{Force reduction} = \frac{F_{eq} - F_T}{F_{eq}} = 1 - \text{TR}$$

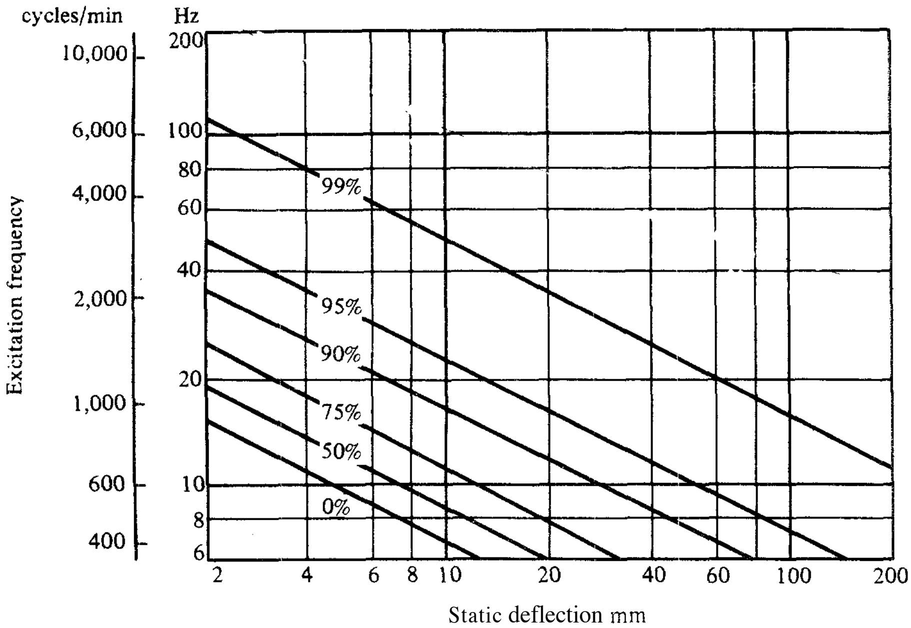


FIG. 3-27. Percentage reduction in force transmitted to foundation in *vibration* isolation, $\zeta = 0$.

where F_{eq} and F_T are the amplitudes of the excitation and the transmitted force, respectively. It is observed in Fig. 3-25 that low natural frequency and low damping are desirable for vibration isolation. Assume $\zeta \simeq 0$ and $r > 1$ in Eq. (3-32). Thus, $TR = 1/(r^2 - 1)$ and the force reduction becomes

$$\text{Force reduction} = \frac{r^2 - 2}{r^2 - 1}$$

Since $r^2 = (\omega/\omega_n)^2$, $\omega_n^2 = k/m$, and the static deflection of a spring $\delta_{st} = mg/k$, the equation above reduces to

$$\text{Force reduction} = \frac{\omega^2 \delta_{st} - 2g}{\omega^2 \delta_{st} - g} \tag{3-34}$$

The equation is plotted in Fig. 3-27

Example 17

An air compressor of 450 kg mass (992 lb,) operates at a constant speed of 1,750 rpm. The rotating parts are well balanced. The reciprocating parts are of 10 kg (22 lb,). The crank radius is 100 mm (4 in.). If the damper for the mounting introduces a damping factor $\zeta = 0.15$, (a) specify the springs for the mounting such that only 20 percent of the unbalance force is transmitted to the foundation, and (b) determine the amplitude of the transmitted force.

Solution:

(a) $\omega = 2\pi(1750/60) = 183.3$ rad/s
Since $\text{TR} = 0.20 = \sqrt{1+(2\zeta r)^2}/\sqrt{(1-r^2)^2+(2\zeta r)^2}$

$$\therefore \quad r = 2.72 = \omega/\omega_n \qquad \text{or} \qquad \omega_n = 183.3/2.72 = 67.4 \text{ rad/s}$$

$$k = m_t\omega_n^2 = 450(67.4)^2 = 2042 \text{ kN/m } (11{,}660 \text{ lb}_f\text{/in.})$$

(b) Amplitude of the force transmitted $= 0.20\, F_{eq}$
$= 0.20 m e \omega^2 = 0.20(10)(0.10)(183.3)^2 = 6.72$ kN (1,510 lb_f)

Case 4. System Attached to Moving Support

When an excitation motion is applied to the support or the base of the system instead of applying to the mass, both the absolute motion of the mass and the relative motion between the mass and the support are of interest. We shall consider the absolute motion of the mass in this case.

Let the base of the system in Fig. 3-23(b) be given a harmonic displacement $x_1(t)$. The corresponding displacement of m is $x_2(t)$. The spring force is $k(x_2 - x_1)$ and the damping force is $c(\dot{x}_2 - \dot{x}_1)$. Applying Newton's second law to the mass m yields

$$m\ddot{x}_2 = -k(x_2 - x_1) - c(\dot{x}_2 - \dot{x}_1) \tag{3-35}$$

which can be rearranged to

$$m\ddot{x}_2 + c\dot{x}_2 + kx_2 = kx_1 + c\dot{x}_1$$

Applying the impedance method gives $x_1 = \bar{X}_1 e^{j\omega t}$. The quantity $(kx_1 + c\dot{x}_1) = (k + j\omega c)\bar{X}_1 e^{j\omega t} = \bar{F}_{eq} e^{j\omega t}$ can be considered as an equivalent force as shown in Eq. (3-22). Hence the equation above reduces to

$$(k - \omega^2 m + j\omega c)\bar{X}_2 = (k + j\omega c)\bar{X}_1$$

or

$$\frac{\bar{X}_2}{\bar{X}_1} = \frac{k + j\omega c}{k - \omega^2 m + j\omega c} = \frac{X_2}{X_1} e^{j(\gamma - \phi)} \tag{3-36}$$

where

$$\frac{X_2}{X_1} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \tag{3-37}$$

$$\gamma - \phi = \tan^{-1} 2\zeta r - \tan^{-1}\frac{2\zeta r}{1-r^2} \tag{3-38}$$

and $\omega_n^2 = k/m$, $r = \omega/\omega_n$, and $c/k = 2\zeta/\omega_n$.

Since Eq. (3-37) for X_2/X_1 is identical to Eq. (3-32) for F_T/F_{eq}, both can be called the transmissibility equation, although the former denotes the transmission of motion from the base to the mass and the latter the transmission of force from the mass to its foundation. Hence Fig. 2-25 is a plot of both equations.

Example 18. Mounting of Instruments

An instrument of mass m is mounted on a vibrating table as shown in Fig. 3-23(b). Find (a) the maximum acceleration of the instrument, and (b) the maximum force transmitted to the instrument. Assume the motion $x_1(t)$ of the table is harmonic at the frequency w.

Solution:

(a) The equation of the mass m is as shown in Eq. *(3-35)*. The maximum acceleration of m is $\ddot{x}_{2\max} = \omega^2 X_2$. Applying Eq. *(3-37)* yields

$$\frac{\ddot{x}_{2\max}}{X_1} = \frac{\omega^2 X_2}{X_1} = \frac{\omega^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

where $r = \omega/\omega_n$ and $\omega_n^2 = k/m$. Alternatively, we can compare the acceleration of m with that of the table, that is,

$$\frac{\ddot{x}_{2\max}}{\ddot{x}_{1\max}} = \frac{\omega^2 X_2}{\omega^2 X_1} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

Thus, the characteristics of the acceleration ratio is the same as the displacement ratio X_2/X_1 and plotted in Fig. *3-25*.

(b) Force is transmitted to m through the spring and the damper. From Eq. (3-35), the sum of these forces is $m\ddot{x}_{2\max} = mw^2X_2$. Applying Eq. (3-37), the maximum force transmitted $F_{T\max}$ is

$$\frac{F_{T\max}}{X_1} = m\omega^2\frac{X_2}{X_1}$$

Comparing $F_{T\max}$ with the maximum acceleration of the support, we have $\ddot{x}_{1\max} = \omega^2 X_1$ and

$$\frac{F_{T\max}}{\ddot{x}_{1\max}} = m\frac{X_2}{X_1} = \frac{m\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

Hence, with the exception of the constant m, this equation is represented in Fig. *3-25*. Comparing $F_{T\max}$ with the maximum displacement of the support, we have

$$\frac{F_{T\max}}{X_1} = kr^2\frac{X_2}{X_1} = \frac{kr^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

where $mw^2 = k(\omega/\omega_n)^2 = kr^2$. Hence, with the exception of the constant k, this equation is represented in Fig. *3-26*.

Example 19. Vehicle Suspension

A vehicle is a complex system with many degrees of freedom. As a first approximation, Fig. *3-28* may be considered as a vehicle driven on a rough

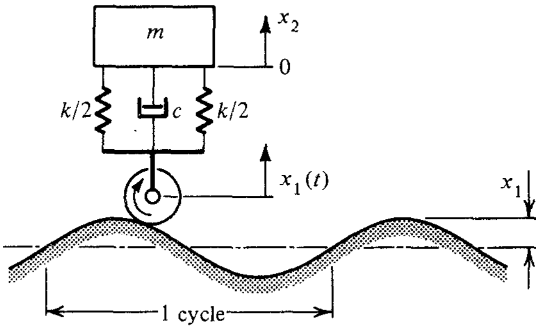


FIG. 3-28. Schematic sketch of vehicle moving over rough road.

road. It is assumed that (1) the vehicle is constrained to one degree of freedom in the vertical direction, (2) the spring constant of the tires is infinite, that is, the road roughness is transmitted directly to the suspension system of the vehicle, and (3) the tires do not leave the road surface. Assume a trailer has 1,000 kg mass (2,200 lb,) fully loaded and 250 kg empty. The spring of the suspension is of 350 kN/m (2,000 lb_f/in.). The damping factor $\zeta = 0.50$ when the trailer is fully loaded. The speed is 100 km/hour (62 mph). The road varies sinusoidally with 5.0 mlcycle (16.4 ft/cycle). Determine the amplitude ratio of the trailer when fully loaded and empty.

Solution:

The excitation frequency is

$$f = \frac{100,000}{3,600} \times \frac{1}{5} = 5.56 \text{ Hz}$$

From Eq. (2-27), the damping coefficient is $c = 2\zeta\sqrt{km}$. Since c and k are constant, ζ varies inversely with the square root of m. Thus, $\zeta = 1.0$ when the trailer is empty. Applying Eq. (3-37), the calculations are tabulated as follows:

ITEM	TRAILER, FULLY LOADED	TRAILER, EMPTY
Natural frequency		
$\omega_n = \sqrt{k/m}$	$\omega_n = \sqrt{(350,000)/1000}$	$\omega_n = 2(18.7)$
	$= 18.7$ rad/s	$= 37.4$ rad/s
	$= 2.98$ Hz	$= 5.96$ Hz
$r = \omega/\omega_n$	$r = 5.56/2.98$	$r = 5.56/5.96$
	$= 1.87$	$= 0.93$
Ratio of X_2/X_1		
$= \dfrac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$	$X_2 = 2.12X_1/3.10$	$X_2 = 2.12X_1/1.87$
	$= 0.68X_1$	$= 1.13X_1$

The amplitude ratio when fully loaded and empty is $0.68/1.13 = 111.67$.

Case 5. Seismic Instruments

A schematic sketch of a seismic instrument is shown in Fig. 3-29. It consists of a mass m attached to the base by means of springs and dampers, that is, the base of the instrument is securely attached to a vibrating body, the motion of which is to be measured. Let $x_1(t)$ be the motion of the base and $x_2(t)$ the motion of m. The relative motion $(x_2 - x_1)$ is used to indicate $x_1(t)$. As illustrated, $(x_2 - x_1)$ is recorded by means of a pen and a rotating drum.

Since this system is essentially the same as that shown in Fig. 3-23(b), its equation of motion is identical to Eq. (3-35). Let $x,= X,\sin \omega t$. Defining $x(t) = x_2(t) - x_1(t)$ and substituting $\ddot{x}_2(t) = \ddot{x}(t) + \ddot{x}_1(t)$ into Eq. *(3-35)* yields

$$mf + c\dot{x} + kx = -m\ddot{x}_1 = m\omega^2 X_1 \sin ot \qquad \textit{(3-39)}$$

which is of the same form as Eq. *(3-22)* with $F_{eq} = m\omega^2 X_1$.

Using the impedance method in Sec. *2-6,* Eq. *(3-39)* gives

$$\bar{X} = \frac{0^2 m}{k - 0^2 m + j\omega c} X_1 - Xe^{-j\phi}$$

where $\phi = \tan^{-1} 2\zeta r/(1 - r^2)$, $\omega_n^2 = k/m$, and $r = \omega/\omega_n$. The: amplitude ratio X/X_1 from the equation above is

$$\frac{X}{X_1} = r^2 R = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \qquad \textbf{(3-40)}$$

The right side of this equation is identical to Eqs. *(3-28)* and (3-30). Hence the characteristic of the equation is also portrayed in Fig. ***3-16.***

The relative motion between the mass m and its base in a seismic instrument, or vibration pickup, can be measured mechanically as illustrated in Fig. ***3-29.*** For high speed operations and convenience, this motion is often converted into an electrical signal. The schemes illustrated in Figs. 3-30(a) and *(b)* for this conversion are self-evident. **A** typical piezoelectric accelerometer is illustrated in Fig. 3-30(c). The piezoelectric

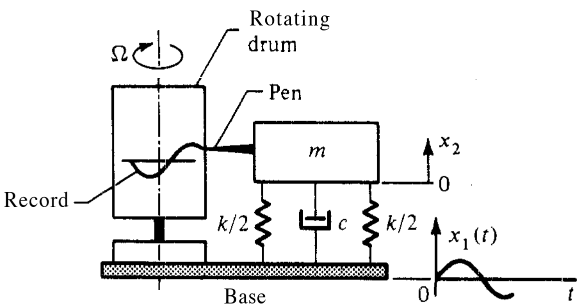
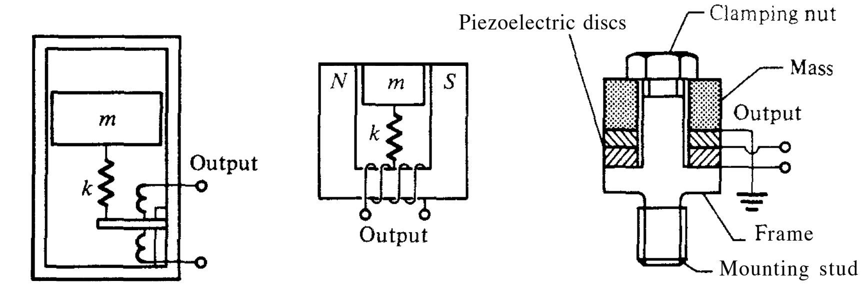


FIG. 3-29. *Schematic sketch of seismic instrument.*



(a) Pickup with strain sensitive element (b) Variable-reluctance pickup (c) Piezoelectric accelerometer

FIG. 3-30. *Vibration pickups with electrical output.*

elements are sandwiched between the mass and the frame. The voltage output of the device is due to the cyclic deformation of the piezoelectric crystals. The effective spring, damping, and mass in this accelerometer, however, are not self-evident. Since vibration measurement is a separate study, we shall not pursue the subject further.

Vibrometer: Referring to Fig. 3-16, if $\omega \gg \omega_n$ or $r \gg 1$, the ratio X/X_1 in Eq. (3-40) approaches unity regardless of the value of ζ. In other words, the relative displacement $x(t)$ is equal to the displacement $x_1(t)$, which is the motion to be measured. The phase angle is approximately 180°. An instrument for displacement measurement is called a *vibrometer*. Since there is no advantage in introducing damping in the system, a vibrometer is designed with damping only to minimize the transient vibration.

The performance characteristics of a typical inductive type velocity pickup is illustrated in Fig. 3-31. The useful range is marked off in the figure with bold lines. Displacement and acceleration can be obtained from the electrical output of the velocity pickup by integration and differentiation. The natural frequency of this pickup is 8 Hz. The displacement amplitude ranges from 0.0025 to 10 mm.

Accelerometer: A seismic instrument to measure acceleration is called an *accelerometer*. Due to their small size and high sensitivity, most vibration measurements today are made with accelerometers. The velocity and displacement can be obtained from the electrical output of the accelerometer by integration. For example, the motion of the piston of an internal combustion engine can be indicated in this manner.

If the motion to be measured is $x_1 = X_1 \sin \omega t$, the amplitude of the acceleration is $\omega^2 X_1$. From Eq. (3-40), we obtain

$$X = \frac{R}{\omega_n^2}(\omega^2 X_1) = \frac{1}{\omega_n^2\sqrt{(1-r^2)^2+(2\zeta r)^2}}\,\omega^2 X_1 \tag{3-41}$$

The quantity ω_n^2 is a constant, since ω_n is a property of the system. The relative motion $x(t)$ is proportional to the acceleration $\ddot{x}_1(t)$ if the magnification factor R is constant for all ranges of operation.

A periodic vibration generally has a number of harmonic components, each of which gives a corresponding value of r $(=\omega/\omega_n)$ in Eq. (3-41). ***Amplitude distortion*** occurs if the magnification factor $1/\sqrt{(1-r^2)^2+(2\zeta r)^2}$ changes with the harmonic components. In other words, the magnification R of each harmonic component must be identical in order to reproduce the input waveform. Since $R \simeq 1$ when r approaches zero, an accelerometer is constructed such that $r \ll 1$, or $\omega_n \gg \omega$. The percent amplitude distortion is defined as

$$\text{Amplitude distortion} = (R-1)\times 100\% \tag{3-42}$$

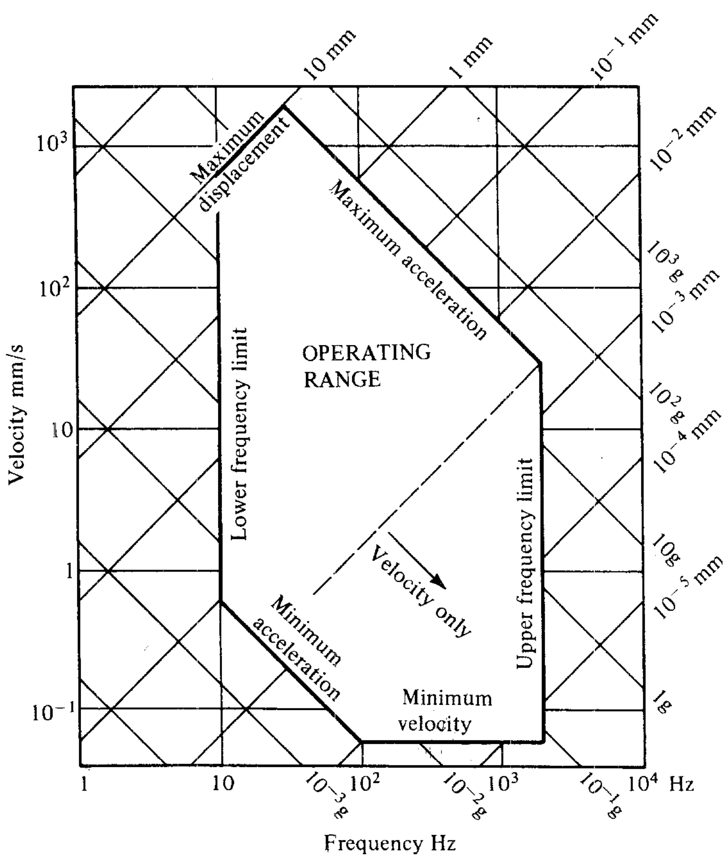


FIG. 3-31. *Performance characteristics of a typical inductive velocity pickup.*

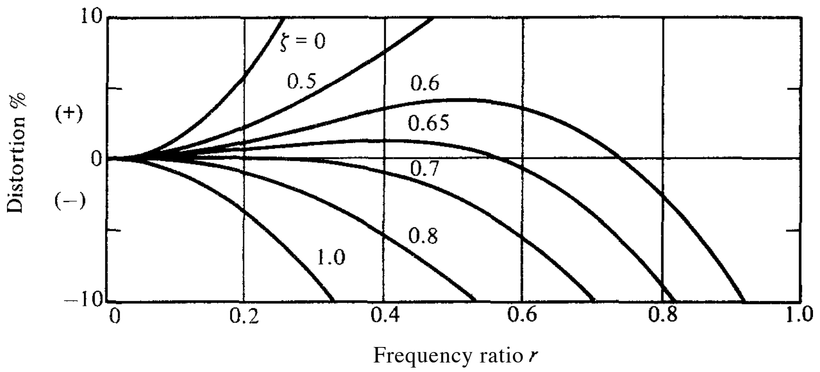


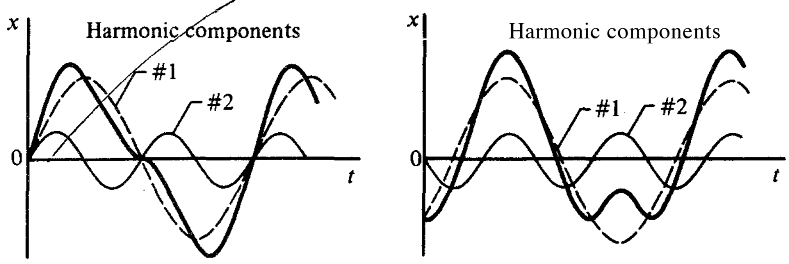
FIG. 3-32. *Amplitude distortion in accelerometer.*

This is plotted in Fig. 3-32. Note that (1)an accelerometer should be built with $0.6<\zeta<0.7$ in order to minimize the amplitude distortion and (2) the usable range is $0<r<0.6$.

Phase distortion occurs if there is a shift in the relative phase between the harmonic components in a periodic signal. Assume a periodic signal $x(t)$ in Fig. 3-33(*a*) has two harmonic components. The recorded signal in Fig. 3-33(*b*) consists of the same components without amplitude distortion. The relative phase between the components, however, has changed. Evidently the distortion in the wave-form of the recorded signal is due to the phase distortion. Phase distortion is secondary for some applications. It is important for the applications in which the wave-form must be preserved.

For zero phase distortion, the phase shift ϕ of each of the harmonic components in the signal must increase linearly with frequency. Consider the equations

$$\begin{aligned} x(t) &= x_1(t)+x_2(t) \\ &= X_1 \sin(\omega_1 t - 4,) + X_2 \sin(\omega t - \phi_2) \\ &= X_1 \sin \omega_1(t-\phi_1/\omega_1) + X_2 \sin \omega_2(t-\phi_2/\omega_2) \\ &= X_1 \sin \omega_1(t-t_{\phi 1}) + X_2 \sin \omega_2(t-t_{\phi 2}) \end{aligned}$$



(a) Original signal (*b*) Recorded signal

FIG. 3-33. *Phase distortion.*

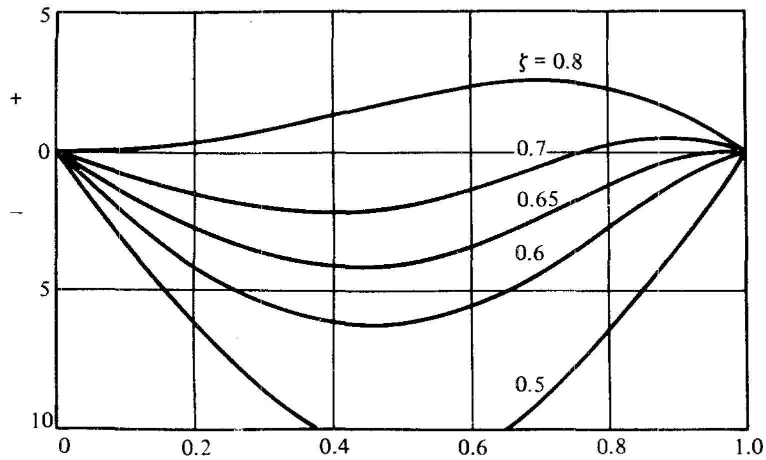


FIG. 3-34. *Phase distortion in accelerometer.*

The quantities $t_{\phi 1}$ and $t_{\phi 2}$ denote the delay or the shift of the signal along the positive time axis. If each harmonic component of a signal $x(t)$ is shifted by the same amount of time, the wave-form is preserved. This requires that $t_{\phi_1} = t_{\phi_2} = \text{constant}$, that is, $\phi_1/\omega_1 = \phi_2/\omega_2 = \text{constant}$. In other words, the phase angle varies linearly with the frequency ω.

It is observed in Fig. 2-9 that, for the phase angle ϕ to vary linearly with frequency over an acceptable range $0<r<1$, the phase shift is $(90°\times r)$. Hence the phase distortion of an accelerometer is defined as

$$\text{Phase distortion} = (\phi - 90r)\,\text{deg} \qquad (3\text{-}43)$$

This is plotted in Fig. 3-34. Again, it is seen that an appropriate damping in an accelerometer is necessary in order to minimize the phase distortion.

Example 20

A machine component is vibrating with the motion

$$\begin{aligned} Y &= y_1(t) + y_2(t) \\ &= Y_1 \sin \omega_1 t + Y_2 \sin \omega_2 t \\ &= 0.10 \sin 60\pi t + 0.05 \sin 120\pi t \end{aligned}$$

Determine the vibration record that would be obtained with an accelerometer. Assume $\zeta = 0.65$ and $f_n = \omega_n/2\pi = 1{,}500\ Hz$.

Solution:

From Eq. (3-39), the equation of motion is

$$m\ddot{x} + c\dot{x} + kx = -m\ddot{y} = -m(\ddot{y}_1 + \ddot{y}_2)$$

The harmonic response due to each of the components in the input can be obtained from Eq. (3-40). By superposition, we have

$$x = r_1^2 R_1 Y_1 \sin(\omega_1 t - \phi_1) + r_2^2 R_2 Y_2 \sin(\omega_2 t - \phi_2)$$

where

$$\phi_1 = \tan^{-1} \frac{2\zeta r_1}{1 - r_1^2} \qquad \text{and} \qquad \phi_2 = \tan^{-1} \frac{2\zeta r_2}{1 - r_2^2}$$

The frequency ratios are $r_1 = \omega_1/\omega_n = 60\pi/3{,}000\pi = 1/50$ and $r_2 = 120\pi/3{,}000\pi = 2/50$. The values of the magnification factors R_1 and R_2 are almost unity. Thus,

$$r_1^2 R_1 Y_1 = (1/50)^2(1)(0.10) = 0.10/2500$$

$$r_2^2 R_2 Y_2 = (2/50)^2(1)(0.05) = 0.20/2500$$

$$\phi_1 = \tan^{-1} \frac{(2)(0.65)(1/50)}{1-(1/50)^2} = \tan^{-1} 0.026 = 1.49°$$

$$\phi_2 = \tan^{-1} \frac{(2)(0.65)(2/50)}{1-(2/50)^2} = \tan^{-1} 0.052 = 2.98°$$

Hence the acceleration record is

$$x = \frac{1}{2500}[0.10 \sin(60\pi t - 1.49°) + 0.20 \sin(120\pi t - 2.98°)]$$

The output of an accelerometer is usually converted to an electrical signal and amplified. Hence the value of the recorded acceleration depends on the amplification used in the data processing. If the equation above is integrated twice to give a displacement, the measured value of the given motion is

$$\underset{(\text{measured})}{y} = (\text{constant})[0.10 \sin(60\pi t - 1.49°) + 0.05 \sin(120\pi t - 2.98°)]$$

Note that the measured value of $y(t)$ has practically no amplitude distortion and only a slight shift in the phase angles. There is no phase distortion, however, because the phase shift is linear with the frequency of the harmonic components. There is a slight time delay between the input $y(t)$ and its measured value. The time delay is ϕ/ω. For the given values, we have time delay $= 1.49° (\pi/180°)/(60\pi) = 0.14$ ms. The results above are due to the high natural frequency of the accelerometer.

Case 6. Elastically Supported Damped Systems

The damping of a real system can be considerably more complex than a simple damper shown in Fig. **3-23.** Equivalent viscous damping will be discussed in Sec. **3-8.** We shall consider the elastically supported damper as illustrated in Fig. 3-35.

From the free body sketch, the equation of motion for the mass m is

$$m\ddot{x} + c(\dot{x} - \dot{x}_1) + kx = F \sin \omega t$$

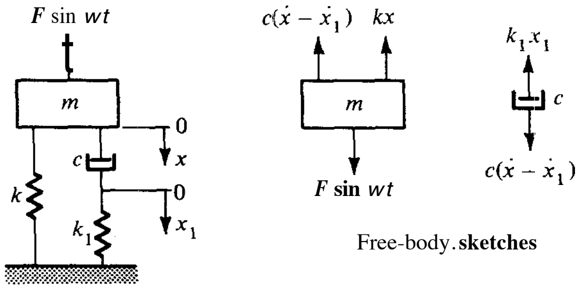


FIG. 3-35. Elastically supported damper.

Since the damper c and the spring k_1 are in series, the damping force is equal to the spring force.

$$c(\dot{x} - \dot{x}_1) = k_1 x_1$$

Since the motions $x(t)$ and $x_1(t)$ are harmonic, the equations above can be solved readily by the mechanical impedance method cliscussed in Sec. 2-6. Substituting $\bar{F}e^{j\omega t}$ for Fsin ωt, $j\omega$ for the time derivatives, and using phasor notation, the equations become

$$\begin{aligned}(k - \omega^2 m + j\omega c)\bar{X} - j\omega c\bar{X}_1 &= \bar{F} \\ -j\omega c\bar{X} + (k_1 + j\omega c)\bar{X}_1 &= 0\end{aligned} \tag{3-44}$$

The phasors $\bar{X}$ and $\bar{X}_1$ can be solved for by Cramer's rule:

$$\begin{aligned}\bar{X} &= \frac{\bar{F}(k_1 + j\omega c)}{k_1(k - \omega^2 m) + j\omega c(k + k_1 - \omega^2 m)} = Xe^{-j\gamma} \\ \bar{X}_1 &= \frac{j\omega c\bar{F}}{k_1(k - \omega^2 m) + j\omega c(k + k_1 - \omega^2 m)} = X_1 e^{-j\gamma_1}\end{aligned} \tag{3-45}$$

Defining the stiffness ratio of the springs as $N = k_1/k$, $\omega_n = \sqrt{k/m}$, $c/m = 2\zeta\omega_n$, $r = \omega/\omega_n$, and

$$\begin{aligned}\Delta(\omega) &= |k_1(k - \omega^2 m) + j\omega c(k + k_1 - \omega^2 m)|/(kk_1) \\ &= \sqrt{(1 - r^2)^2 + [2\zeta r(1 + 1/N - r^2/N)]^2}\end{aligned} \tag{3-46}$$

we obtain

$$\begin{aligned}X &= \frac{F}{k}\sqrt{1 + (2\zeta r/N)^2}/\Delta(\omega) \\ X_1 &= \frac{F}{k}(2\zeta r/N)/\Delta(\omega)\end{aligned} \tag{3-47}$$

$$\begin{aligned}\gamma &= \tan^{-1}\frac{2\zeta r(1 + 1/N - r^2/N)}{1 - r^2} - \tan^{-1}\frac{2\zeta r}{N} \\ \gamma_1 &= \tan^{-1}\frac{2\zeta r(1 + 1/N - r^2/N)}{1 - r^2} - \frac{\pi}{2}\end{aligned} \tag{3-48}$$

*Example 21. Vibration Isolation**

A machine of mass m is mounted on a spring-damper system as shown in Fig. *3-35*. Define $\omega_n = \sqrt{k/m}$, $\zeta = c/2\sqrt{km}$, $r = \omega/\omega_n$, and $N = k_1/k$: *(a)* Derive the equation for the transmissibility TR of the system. (b) If $r = 0.6$, $N = 2$, and $\zeta = 0.4$, find the transmissibility TR. (c) Repeat part b if $r = 10$ and the other parameters remained unchanged. **(d)** Compare the values of TR from part b and part **c** with that expressed in Eq. *(3-32)*.

Solution:

(a) The force transmitted F_T to the foundation is the sum of the forces transmitted through the springs k and k_1.

$$F_T = kx + k_1 x_1$$

The phasors $\bar{X}$ and $\bar{X}_1$ are given in Eq. (3-45). Using vectorial addition, the phasor $\bar{F}_T$ of the transmitted force is

$$\bar{F}_T = kZ + k_1\bar{X}_1$$

$$= \frac{F[k(k_1 + jwc) + j\omega k_1 c]}{k_1(k - \omega^2 m) + j\omega c(k + k_1 - \omega^2 m)}$$

or

$$\bar{F}_T = \frac{F\sqrt{1 + [2\zeta r(1 + 1/N)]^2}}{\Delta(\omega)} e^{-j\gamma_T} \tag{3-49}$$

where γ_T is the phase angle of the transmitted force relative to the excitation and $\Delta(\omega)$ is defined in Eq. *(3-46)*. The transmissibility TR is the ratio of the magnitude of the transmitted force relative to that of the excitation; that is,

$$\text{TR} = F_T/F = \sqrt{1 + [2\zeta r(1 + 1/N)]^2}/\Delta(\omega)$$

(b) For $r = 0.6$, $N = 2$, and $\zeta = 0.4$, we obtain

$$\text{TR} = 1.23/0.90 = 1.37$$

(c) For $r = 10$, $N = 2$, and $\zeta = 0.4$, we have

$$\text{TR} = 12/400 = 0.03$$

(d) Substituting the corresponding values in Eq. (3-32), the values of TR for the system in Fig. 3-21*(a)* are

$$\text{TR} = \sqrt{1.23/0.64} = 1.39 \qquad \text{for} \qquad r = 0.6$$

$$TR = \sqrt{65/9865} = 0.081 \qquad \text{for} \qquad r - 10$$

* **A detailed** analysis is shown in J. *C*. Snowdon, ***Vibration and Shock in Damped Mechanical Systems,*** John Wiley & Sons, Inc., New York, 1968, pp. **33-38.**

Comparing parts (b), (c), and (d), the transmissibility for the two isolation systems are approximately the same for r = 0.6. When operating at r = 10, the system in Fig. 3-35 seems to be superior to that in Fig. 3-23(a).

3-6 DAMPED FORCED VIBRATION—PERIODIC EXCITATION

Harmonic response of systems was presented in the last section. Forces arising from machinery are commonly periodic but seldom harmonic. In considering periodic excitations, we are in effect generalizing the previous applications. We shall briefly review the Fourier series and then illustrate the applications.

As illustrated in Sec. 1-2, a function is periodic if

$$F(t) = F(t \pm \tau) \qquad (3\text{-}50)$$

where τ is the period, or the minimum time required for $F(t)$ to repeat itself. The Fourier series* expansion of $F(t)$ is

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (\text{a, cos not} + b_n \text{ sin not}) \qquad (3\text{-}51)$$

where n is a positive integer, a, and b_n are the coefficients of the infinite series. Note that $a_0/2$ gives the average value of $F(t)$. The fundamental frequency of the periodic function is o $= 2\pi/\tau$, that is, when n = 1. The frequency of the nth harmonic is no $= 2n\pi/\tau$, that is, when n > 1.

The following relations are used to evaluate a, and b_n:

$$\int_0^\tau \cos m\omega t \cos \text{nor}\, dt = \begin{cases} 0 & \text{if} \quad m \neq n \\ \tau/2 & \text{if} \quad m = n \end{cases}$$

$$[\text{sin} \quad \text{mot sin not}\, dt = \begin{cases} 0 & \text{if} \quad m \neq n \\ \tau/2 & \text{if} \quad m = n \end{cases} \qquad (3\text{-}52)$$

$$\int_0^\tau \text{cos mot sin not}\, dt = 0 \qquad \text{whatever m and n}$$

where m and n are integers and $\tau = 2\pi/\omega$ is the period of $F(t)$. Rewriting the series in an expanded form, we get

$$F(t) = \left(\frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + \cdots\right) + (\text{b, } \sin \omega t + b_2 \sin 2\omega t + \cdots)$$

A particular coefficient a, can be obtained by multiplying both sides of

*See, for example, I. S. Sokolnikoff and R. M. Redheffer, ***Mathematics*** of ***Physics and Modern Engineering,*** McGraw-Hill Book Co., New York, 1958, p. 175.

this equation by (cos $p\omega t$) and integrating each term using the relations in Eq. (3-52). Except for the term containing a„ all the integrals on the right side are identically zero. Thus,

$$\int_0^\tau F(t)\cos \text{pot}\, dt = 0 + \cdots + 0 + \int_0^\tau a_p \cos^2 p\omega t\, dt + 0 + \cdots$$
$$= \frac{a_p \tau}{2}$$

or

$$a_p = \frac{2}{\tau}\int_0^\tau F(t)\cos p\omega t\, dt$$

Similarly, a particular coefficient b_p can be obtained by multiplying the series by (sin pot) and applying the relations in Eq. (3-52). Thus, the coefficients of the Fourier series in Eq. (3-51) are

$$a_0 = \frac{2}{\tau}\int_0^\tau F(t)\, dt$$
$$a_n = \frac{2}{\tau}\int_0^\tau F(t)\cos \text{nor}\, \text{dt} \qquad \textbf{(3-53)}$$
$$b_n = \frac{2}{\tau}\int_0^\tau F(t)\sin n\omega t\, dt$$

Let a periodic force $F(t)$ be applied to a one-degree-of-freedom system. $F(t)$ may represent the equivalent force in any of the five cases enumerated in the previous section. Expanding $F(t)$ in a Fourier series and applying Eq. (3-22), the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \text{not} + b_n \sin \text{not}) \qquad (3\text{-}54)$$

The steady-state response due to each of the components of the excitation can be calculated. By superposition, the steady-state response of the system is

$$x = \frac{a_0}{2\text{k}} + \sum_{n=1}^{\infty} \frac{\text{a,} \cos(n\omega t - \phi_n) + b_n \sin(n\omega t - \phi_n)}{k\sqrt{(1-n^2r^2)^2 + (2\zeta nr)^2}} \qquad (3\text{-}55)$$

where

$$\phi_n = \tan^{-1}\frac{2\zeta nr}{1-\text{n}^z\text{r}^2} \qquad \text{and} \qquad r = \frac{\omega}{\omega_n}$$

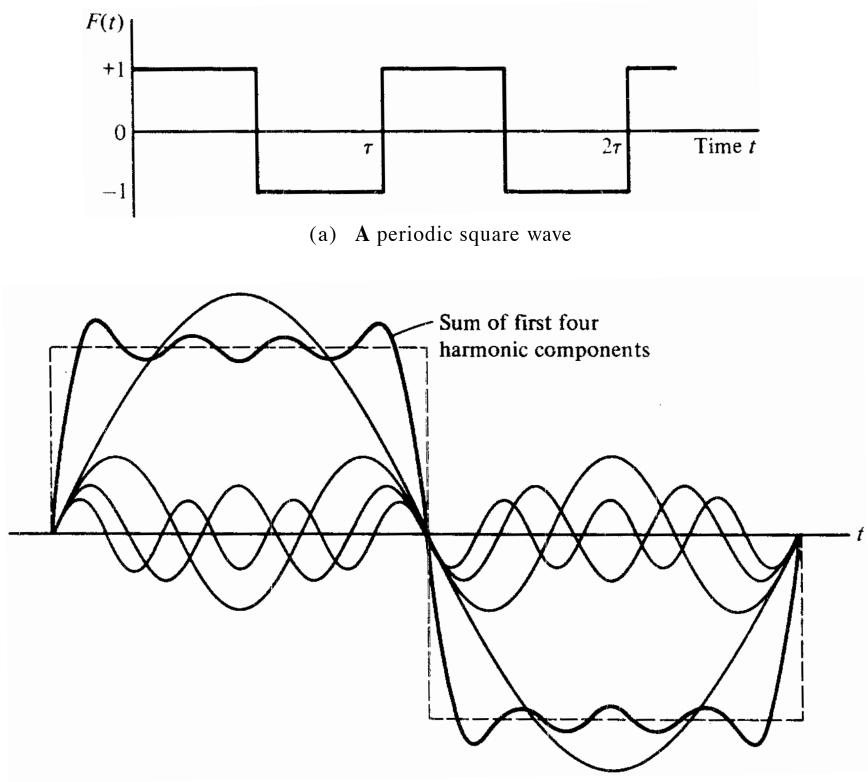
Example *22*

Find the Fourier series of the square wave in Fig. 3-36(a).

Solution:

For any one cycle, the given periodic function is

$$F(t) = \begin{cases} 1 & \text{for} \quad 0 < t < \tau/2 \\ -1 & \text{for} \quad \tau/2 < t < \tau \end{cases}$$



(a) A periodic square wave

(b) Harmonic components of a square wave

FIG. 3-36. Fourier series analysis of a square wave.

Applying Eq. (3-53) and for $\omega = 2\pi/\tau$, the coefficients of the Fourier series of $F(t)$ are

$$a_0 = \frac{2}{\tau}\int_0^{\tau} F(t)\,dt = \frac{2}{\tau}\left[\int_0^{\tau/2} (1)\,dt - \int_{\tau/2}^{\tau} (1)\,dt\right] = 0$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} F(t)\cos n\omega t\,dt$$

$$= \frac{2}{\tau}\left[\int_0^{\tau/2} \cos n\omega t\,dt - \int_{\tau/2}^{\tau} \cos n\omega t\,dt\right] = 0$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} F(t)\sin n\omega t\,dt$$

$$= -\frac{2}{\tau}\frac{\tau}{2n\pi}\left(\cos\frac{2n\pi}{\tau}t\Big|_0^{\tau/2} - \cos\frac{2n\pi}{\tau}t\Big|_{\tau/2}^{\tau}\right)$$

$$= \begin{cases} \dfrac{4}{n\pi} & \text{for n odd} \\ 0 & \text{for n even} \end{cases}$$

Hence the Fourier series expansion of the square wave is

$$F(t)=\frac{4}{\pi}\sum_{n}\frac{1}{n}\sin\frac{2n\pi}{\tau}t \qquad \text{for} \qquad n=1,3,5,$$

The first four harmonics of $F(t)$ and their sum are plotted in Fig. 3-36(b).

The impedance method in Sec. 2-6 can be applied readily to this type of problem. We shall consider (1) the Fourier spectrum of the periodic excitation $F(t)$, (2) the transfer function of the system, and then (3) the technique to combine the two spectra to obtain the spectrum of the response. This general technique is applicable to any linear system.

Consider the two terms in Eq. (3-51) of the same frequency no. Their sum can be expressed as

$$a_n \cos \text{nwt} + \text{b,} \sin \text{not} = c_n \cos(n\omega t - a,) \tag{3-56}$$

where

$$c_n=\sqrt{a_n^2+b_n^2} \qquad \text{and} \qquad \alpha_n=\tan^{-1} b_n/a_n \tag{3-57}$$

Note that (1) c_n is the amplitude and a, the phase angle of the excitation at the frequency $n\omega$ and (2) when n = 0, we have $c_n=a_0/2$ and $\alpha_n=0$. Thus, using the vectorial notation, a periodic excitation $F(t)$ can be expressed as

$$F(t)=\sum_{n=0}^{\infty} c_n e^{j(n\omega t-\alpha_n)}=\sum_{n=0}^{\infty}\bar{c}_n e^{jn\omega t} \tag{3-58}$$

where $\bar{c}_n=c_n e^{-j\alpha_n}$ is the phasor of the harmonic component at the frequency no. The plot of c, versus frequency is called the frequency spectrum and α_n versus frequency the phase spectrum of $F(t)$. The two plots as shown in Fig. 3-37(a) are known as the Fourier spectrum.

From Eq. (2-57), the sinusoidal transfer function of a one-degree-of-freedom system is

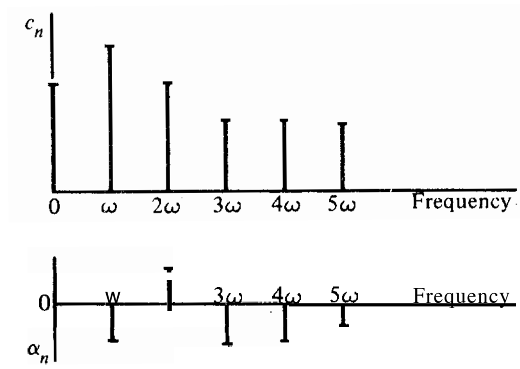
$$\frac{X}{F}(j\omega)=\frac{1}{k-\omega^2 m+j\omega c}=\left|\frac{X}{F}\right| e^{-j\phi} \tag{3-59}$$

The plot of $|X/F|$ versus frequency is analogous to the frequency spectrum and ϕ versus frequency the phase spectrum. In other words, the transfer function plots in Fig. 3-37(b) are the continuous plots of the Fourier spectrum of the system for excitations of unit magnitude and zero phase angle for all frequencies.

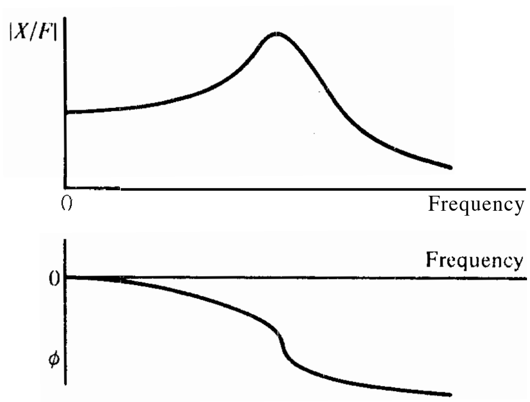
The equation of motion of the system is obtained by substituting Eq. (3-58) into (3-22). Thus,

$$m\ddot{x}+c\dot{x}+kx=\sum_{n=0}^{\infty}\bar{c}_n e^{jn\omega t} \tag{3-60}$$

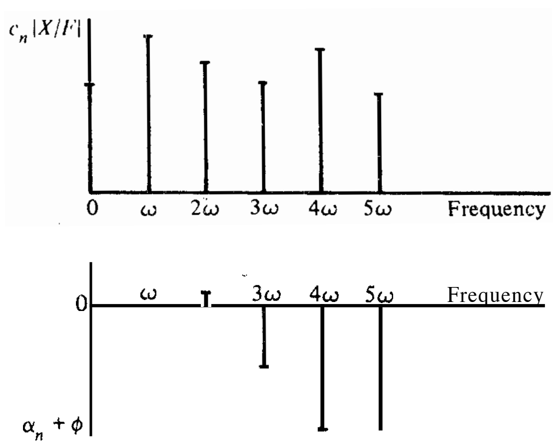
Using the impedance method and Eqs. (3-24) to (3-26), the response due



(a) Fourier spectrum of periodic input $F(t)$



(b) System transfer function



(c) Fourier spectrum of system response

Fig. 3-37. ***Construction of response spectrum from input spectrum and transfer function.***

to a typical component, $c_n = \bar{c}_n e^{jn\omega t} = c_n e^{j(n\omega t - \alpha_n)}$, of the excitation $F(t)$ at the frequency $n\omega$ is

$$x_n = \bar{X}_n e^{jn\omega t} = X_n e^{j(n\omega t - \alpha_n - \phi_n)} \tag{3-61}$$

where

$$\bar{X}_n = (\bar{c}_n)\left(\frac{1}{k - n^2\omega^2 m + jn\omega c}\right)$$

$$= \frac{c_n}{|k - n^2\omega^2 m + jn\omega c|} e^{-j(\alpha_n + \phi_n)} \tag{3-62}$$

and

$$\phi_n = \tan^{-1} \frac{jn\omega c}{k - n^2\omega^2 m} \tag{3-63}$$

Note that the phasor $\bar{X}_n$ of a harmonic response in Eq. (3-62) is the product of $\bar{c}_n$ and $1/(k - n^2\omega^2 m + jn\omega c)$, both of which are complex numbers at the given frequency $n\omega$. In other words, the Fourier spectrum of the system response is the product of the Fourier spectrum of the excitation $F(t)$ and the system transfer function. The rules for the product of complex numbers are given in Eqs. (1-14) and (1-15), that is, (1) the magnitude of the product is the product of the magnitudes and (2) the phase angle of the product is the algebraic sum of the individual phase angles. At the frequency $n\omega$, the magnitude of the response is $|\bar{c}_n|\,|1/(k - n^2\omega^2 m + jn\omega c)|$, which is the product of the frequency spectrum of $F(t)$ and the magnitude of the system transfer function; the phase angle $-(\alpha_n + \phi_n)$ of the response is the algebraic sum of the phase spectrum of $F(t)$ and the phase angle of the system transfer function. Thus, the Fourier spectrum of the system response can be constructed as shown in Fig. 3-37(c), considering all the harmonic components of $F(t)$.

By the superposition of the individual responses in Eq. (3-61), the system response is

$$x(t) = \sum_{n=0}^{\infty} \bar{X}_n e^{jn\omega t} \tag{3-64}$$

$$x(t) = \sum_{n=0}^{\infty} \frac{c_n}{k\sqrt{(1 - n^2 r^2)^2 + (2\zeta n r)^2}} e^{j(n\omega t - \alpha_n - \phi_n)} \tag{3-65}$$

which is Eq. (3-55) in vectorial form. The waveform of the time response can be constructed from the Fourier spectrum of the response by superposition.

Example 23

A cam actuating a spring-mass system is shown in Fig. 3-38. The total cam lift of the sawtooth is 25 mm (1 in.). The cam speed is 60 rpm. Assume

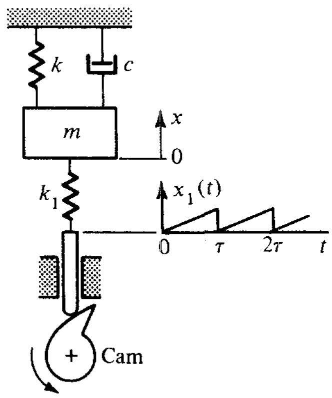


FIG. 3-38. Periodic excitation.

m = 20 kg (44 lb_m) and $k_1 = k = 3.5$ kN/m (20 lb_f/in.). The damping coefficient is c = 0.2 kN . s/m (1.14 lb_f-sec/in.). Find the response $x(t)$.

Solution:

A cycle of the sawtooth motion can be expressed as

$$x_1(t) = \frac{1}{\tau} t \qquad \text{for} \qquad 0 < t < \tau$$

Since the fundamental frequency is 1 Hz or w = 2π, the period τ is 1 sec/cycle. Applying Eq. (3-52), it can be verified readily that

$$a_0 = 1 \qquad a_n = 0 \qquad b_n = -\frac{1}{n\pi}$$

Hence the Fourier series expansion of $x_1(t)$ is

$$x_1(t) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin 2n\pi t$$

The equation of motion of the system is

$$m\ddot{x} + c\dot{x} + (k + k_1)x = k_1 x_1(t) = k_1\left(\frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin 2n\pi t\right)$$

We define $\omega_n^2 = (k + k_1)/m$, $2\zeta\omega_n = c/m$, and $r = \omega/\omega_n$. The response due to the constant excitation term $k_1/2$ is

$$x = \frac{k_1}{2(k + k_1)}$$

The response due to a typical harmonic excitation term at the frequency

$n\omega = 2n\pi$ is as shown in Eq. *(3-62)*.

$$\bar{X}_n = \left(-\frac{k_1}{n\pi}\right)\left(\frac{1}{k_1 + k - n^2\omega^2 m + jn\omega c}\right)$$

$$\bar{X}_n = \frac{-k_1}{n\pi(k_1 + k)\sqrt{(1-n^2r^2)^2 + (2\zeta nr)^2}}\, e^{-j\phi_n}$$

where

$$\phi_n = \tan^{-1}\frac{2\zeta nr}{1 - n^2r^2}$$

By superposition, the total response due to the excitation $F(t)$ is

$$x(t) = \frac{k_1}{k_1 + k}\left(\frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta nr)^2}}\, e^{j(2n\pi t - \phi_n)}\right)$$

For the given data, we have

$$\omega_n^2 = (k_1 + k)/m = (3500 + 3500)/20 = 350 = 18.7^2$$

$$r = \omega/\omega_n = 2\pi/18.7 = 0.107\pi$$

$$\zeta = \tfrac{1}{2}c/\sqrt{(k_1 + k)m} = \tfrac{1}{2}(200)/\sqrt{(3500 + 3500)(20)} = 0.267$$

Thus, the response of the system is

$$x(t) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{\sqrt{[1-(0.11n\pi)^2]^2 + (0.05n\pi)^2}}\sin(2n\pi t - \phi_n)\right)$$

where

$$\phi_n = \tan^{-1}\frac{0.05n\pi}{1 - (0.11n\pi)^2}$$

3-7 TRANSIENT VIBRATION—SHOCK SPECTRUM

The design of equipment to withstand shock is of concern to the engineer. Vibrations induced by the steady-state operation of a machine are generally periodic. This was discussed in the last two sections. Vibrations due to shock and transients usually originate from sources outside the machine or from a sudden change in the machine operation. The transient will die out, but the machine may be damaged or may malfunction momentarily, both of which should be well considered.

A shock is a transient excitation, the duration of which is short compared with the natural period (reciprocal of natural frequency) of oscillation of the system. The transient response due to a transient excitation was discussed in Sec. 2-7. The recording from a vibration test, in the form of a "wavering line" versus time, cannot be used directly by the designer. The shock spectrum is a common method to reduce the test data to a more usable form.

A shock spectrum (response spectrum) is a plot of the peak response versus frequency due to the applied shock. The peak response is that of a number of one-degree-of-freedom systems, each tuned to a different natural frequency. The frequency is that of the natural frequency of the individual systems. The response may be expressed in units of acceleration, velocity, or displacement.* For example, a vibrating reed shown in Fig. 3-3 is a simple mass-spring system. A ***reed gage*** consists of a number of reeds of different natural frequencies. Using a reed gage in a shock test, the maximum displacements of the tips of the reeds give the maximum response for the various natural frequencies. The reed gage can be replaced by a single accelerometer and computers employed to simulate the reeds.† The one-degree-of-freedom system is variously called a ***resonator***, an ***oscillator,*** or a ***simple structure.***

The objective of shock spectrum is to describe the effect of shock rather than the shock itself. Shocks are difficult to characterize and a specific pulse shape is difficult to obtain in a test machine. It is necessary to correlate test data from different laboratories. The shock spectrum is a "common denominator" on the assumption that shocks having the same spectrum would produce similar effect. The spectrum may be regarded as indicative of the potential for damage due to the shock. For example, the peak relative displacement between the mass m and its base for the system in Fig. 3-23(b) is related to the stress in the spring. In other words, the envelope of the spectrum establishes an upper bound of the stress induced, or the damage potential, by a specific shock on the equipment under test.

Types of shock excitation are usually categorized using the undamped resonator as the standard system. Types of shocks and methods of data reduction can be found in the literature.‡ It is paradoxical that a complex study like shock is treated in a seemingly simple manner. The reason is that the time and expense for a detail study must be justified by the past experience of the engineer. An undamped resonator is used, since the largest excursion occurs within the first cycle of the transient and the error introduced by neglecting damping tends to provide a margin of safety. Note that a system under test commonly has more than one degree of freedom. It will be shown in later chapters that a complex system has discrete modes of vibration and a natural frequency is associated with

* C. E. Crede, ***Shock and Vibration Concepts in Engineering Design,*** McGraw-Hill Book Co., New York, 1965, p. 138.

† C. T. Morrow, ***Shock and Vibration Engineering,*** John Wiley and Sons, Inc., New York, 1963, p. 111.

‡ R. S. Ayre, "Transient Response to Step and Pulse Functions," Chap. 8 of ***Shock and Vibration Handbook,*** vol. *1,* C. M. Harris and C. E. Crede (eds.) McGraw-Hill Book, Co., New York, 1961.

S. Rubin, "Concepts in Shock Data Analysis," Chap. 23 of ***Shock and Vibration Handbook,*** vol. 2, C. M. Harris and C. E. Crede (eds.), McGraw-Hill Book Co., New York, 1961.

each of the modes. Hence a complex system can be described in terms of equivalent one-degree-of-freedom systems. Thus, a shock will excite all the modes of a complex system.

If a shock $F_{eq}(t)$ is due to a sudden change in the machine m in Fig. 3-23(a), the equation of motion of m is identical to Eq. (3-1). Assuming zero initial conditions, the response $x(t)$ from Eq. (2-71) is

$$x(t) = \int_0^t F_{eq}(\tau)h(t-\tau)\,d\tau \tag{3-66}$$

where

$$h(t) = \frac{1}{\omega_d m} e^{-\zeta\omega_n t} \sin \omega_d t \tag{3-67}$$

as defined in Eq. (2-69). On the other hand, if the excitation is applied to the base of the machine as shown in Fig. 3-23(b), the equation for the relative displacement $x(t)$ between m and its base is identical to Eq. (3-39).

$$m\ddot{x} + c\dot{x} + kx = -m\ddot{x}_1(t) \tag{3-68}$$

where $x(t) = x_2(t) - x_1(t)$ and $x_1(t)$ and $x_2(t)$ are the absolute motions indicated in the figure. Applying Eq. (2-71) yields

$$x(t) = -\int_0^t m\ddot{x}_1(\tau)h(t-\tau)\,d\tau \tag{3-69}$$

where $h(t)$ is as defined above. The maximum response and the corresponding shock spectrum can be obtained from Eq. (3-66) or (3-69) depending on the application. Computers can be used for the calculations.*

To illustrate a shock spectrum by hand calculation, let a one-half sine pulse $F(t)$ shown in Fig. 3-39 be applied to the mass m of the system in

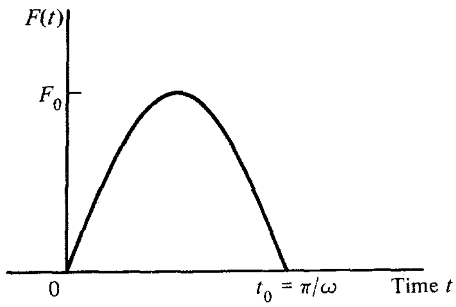


FIG. 3-39. *A* **half-sine pulse.**

* See, for example, J. B. Vernon: ***Linear Vibration and Control System Theory,*** John Wiley & Sons, Inc., New York, 1967, pp. 251.

Fig. 3-23. Assume the system is undamped. $F(t)$ is described by the equation

$$F(t) = \begin{cases} F_0 \sin \omega t & \text{for} \quad 0 \le t \le t_0 \\ 0 & \text{for} \quad t > t_0 \end{cases}$$

For $0 \le t \le t_0$, the system response from Eq. (3-66) is

$$x = \frac{1}{\omega_n m} \int_0^t F_0 \sin \omega\tau \sin \omega_n (t - \tau)\, d\tau$$

which can be integrated to yield

$$x = \frac{F_0}{m(\omega_n^2 - \omega^2)} \left(\sin \omega t - \frac{\omega}{\omega_n} \sin \omega_n t\right) \tag{3-70}$$

Equating $\dot{x}(t) = 0$ to find $x_{\max}$, we get

$$\cos \omega t_m - \cos \omega_n t_m = 0$$

where t_m is the time when $x(t)$ is a maximum or minimum. The roots of this equation is deduced using the identity

$$\cos \omega t_m - \cos \omega_n t_m = -2 \sin \tfrac{1}{2}(\omega + \omega_n) t_m \sin \tfrac{1}{2}(\omega - \omega_n) t_m$$

Thus,

$$t_m = \frac{2n\pi}{\omega_n \pm \omega} \qquad \text{for} \qquad n = \text{integer}$$

Consider $t_m = 2n\pi/(\omega_n + \omega)$. Defining $r = \omega/\omega_n$, the terms in Eq. (3-70) can be expressed as

$$\sin \omega_n t_m = \sin\left(\omega_n \frac{2n\pi}{\omega_n + \omega}\right) = \sin \frac{2n\pi}{1 + r}$$

$$\sin \omega t_m = \sin\left(2n\pi \frac{\omega}{\omega_n + \omega}\right) = \sin\left(2n\pi \frac{r}{1 + r}\right)$$

$$= \sin\left(2n\pi \frac{r + 1 - 1}{1 + r}\right) = \sin 2n\pi\left(1 - \frac{1}{1 + r}\right)$$

$$= -\sin \frac{2n\pi}{1 + r}$$

Hence $\sin \omega_n t_m = -\sin \omega t_m$. Recalling $\omega_n^2 = k/m$, from Eq. (3-70), we get

$$\begin{aligned} x_m &= \frac{F_0}{m(\omega_n^2 - \omega^2)} \left(1\right) - \sin \frac{2n\pi\omega}{\omega_n + \omega} \\ \frac{x_m}{F_0/k} &= \frac{1}{1 - r} \sin \frac{2n\pi r}{1 + r} \end{aligned} \tag{3-71}$$

Similarly, if $t_m = 2n\pi/(\omega_n - \omega)$, we obtain

$$\frac{x_m}{F_0/k} = \frac{1}{1+r}\sin\frac{2n\pi r}{1-r}$$

Comparing the last two equations, it is evident a value of n can be selected to have x_{max} occur at $t_m = 2n\pi/(\omega_n + \omega)$. Equation (3-71) is plotted in Fig. 3-40. The initial shock spectrum, defined by Eq. (3-71), gives the response within the duration of the shock pulse for $0 \le t \le t_0$. Note that there is no solution for $\omega_n/\omega < 1$, since the maximum response does not occur during the pulse if the natural frequency is smaller than the pulse frequency.

For $t > t_0$, the system response from Eq. (3-66) is

$$x = \int_0^{t_0} F(\tau)h(t-\tau)\,d\tau$$

The upper limit of the integration is t_0, because $F(t) = 0$ for $t > t_0$. Performing the integration, substituting $\omega t_0 = \pi$, and simplifying, we get

$$x = \frac{F_0}{2\omega_n m}\left(\frac{\sin\omega_n t - \sin\omega_n(t_0 - t)}{\omega_n + \omega} - \frac{\sin\omega_n t - \sin\omega_n(t_0 - t)}{\omega_n - \omega}\right)$$

It is convenient to define $t' = t - t_0$, a new origin for the time axis. Recalling $\omega_n^2 = k/m$, defining $r = \omega/\omega_n$ and $\omega_n t_0 = \pi/r$, the equation can be

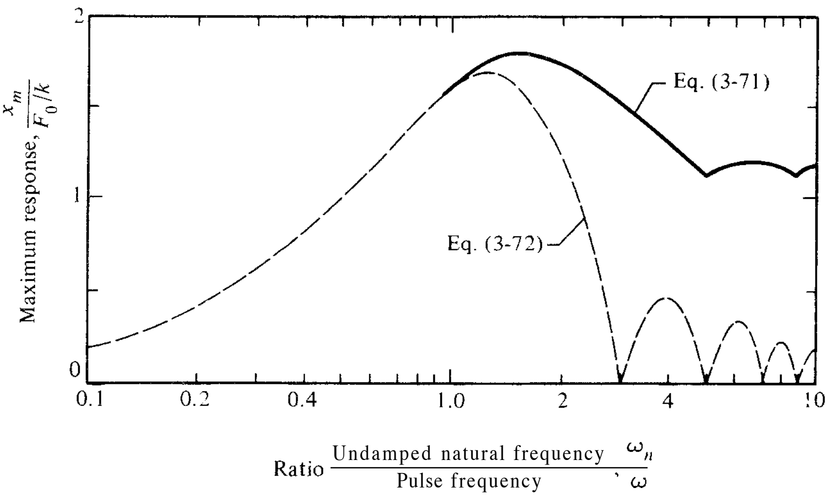


Fig. 3-40. Shock spectra for half-sine pulse applied to m in Fig. 3-23(a): Eq. (3-71) shows initial spectrum; Eq. (3-72) residual spectrum (Crede).

simplified to

$$x = \frac{F_0 r}{k(r^2-1)}\left(\sin\frac{\pi}{r}\cos\omega_n t' + \left(1+\cos\frac{\pi}{r}\right)\sin\omega_n t'\right)$$

The maximum value of $x(t)$ can be expressed as

$$\frac{x_{max}}{F_0/k} = \frac{2r}{r^2-1}\cos\frac{\pi}{2r} \tag{3-72}$$

This gives the residual ,shock spectrum, as it occurs after the shock has terminated. The equation is plotted in Fig. 3-40 and shown as a dash line.*

Now consider the shock spectrum due to a one-half sine pulse $x_1(t)$ applied to the base of the system in Fig. 3-23(b). Assume the system is undamped. The equation of $x_1(t)$ is

$$x_1(t) = \begin{cases} X_1 \sin\omega t & \text{for} \quad 0 \le t \le t_0 \\ 0 & \text{for} \quad t > t_0 \end{cases}$$

For $0 \le t \le t_0$, the equation for the absolute motion $x_2(t)$ is

$$m\ddot{x}_2 + kx_2 = kx_1(t)$$

Since the excitation is $kX_1 \sin \omega t$, the response $x_2(t)$ can be obtained from Eq. (3-70) by substituting kX_1 for F_0, that is,

$$x_2 = \frac{\omega_n^2 X_1}{\omega_n^2 - \omega^2}\left(\sin\omega t - \frac{\omega}{\omega_n}\sin\omega_n t\right) \tag{3-73}$$

Hence the initial shock spectrum is deduced directly from Eq. (3-71):

$$\frac{x_{2max}}{X_1} = \frac{1}{1-r}\sin\frac{2n\pi r}{1+r}$$

The relative motion $x(t)$ between m and its base is given by the relation

$$x = x_2 - x_1$$

where $x_2(t)$ is as defined in **Eq.** (3-73). The expression for $x(t)$ can be maximized to give the corresponding spectrum.

The system is unforced for $t > t_0$. The residual shock spectrum can be calculated as before. The shock spectra for the system shown in Fig. 3-23(b) excited by a half sine pulse is illustrated in Fig. 3-41.†

* C. E. Crede, op. *cit.*, p. 85.

† L. S. Jacobsen and R. S. Ayre, ***Engineering Vibrations,*** McGraw-Hill Book Co., New York, 1958, p. 163.

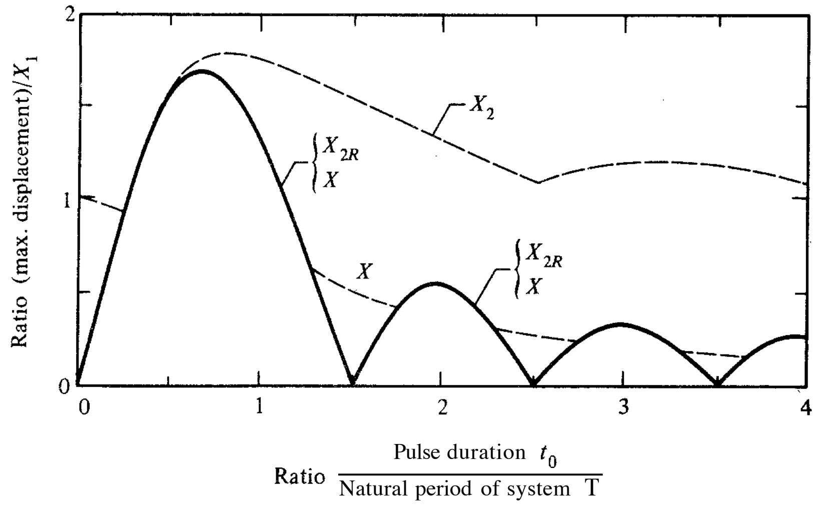


FIG. 3-41 Shock spectra for half-sine pulse applied to the base in Fig. 3-23(b); X is relative displacement; X_2 the initial spectrum; X_{2R} the residual spectrum (Jacobsen & Ayre).

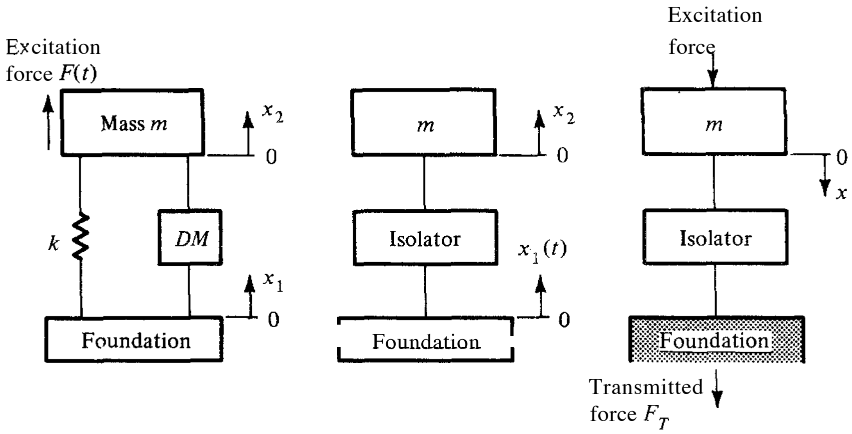
3-8 EQUIVALENT VISCOUS DAMPING

Damping is a complex phenomenon.* It exists whenever there is energy dissipation. Viscous damping was assumed in the previous sections. This occurs only when the velocity between lubricated surfaces is sufficiently low to ensure laminar flow condition. More than one type of damping may exist in a problem. Since damping is generally nonlinear, the superposition of different types of damping in a calculation does not always give reliable results. Furthermore, damping may be dependent on the operating conditions and the past history of the damping mechanism or even the shape of the damper, such as in a visco-elastic material. In order to have a simple mathematical model, we shall examine the viscous equivalent of different types of damping.

Most mechanical systems are inherently lightly damped. The effect of damping may be insignificant for some problems, which may be treated as undamped except near resonance. Damping must be considered, however, in order to control (1) the near resonance conditions of a dynamic system, and (2) the performance of a machine, such as an accelerometer or the riding quality of an automobile. The control can be achieved by using (1) an energy transfer mechanism, such as a dynamic absorber

*The interest and knowledge in damping has increased exponentially in recent years. Between 1945 and 1965, two thousand papers were published in this area.

B. J. Lazan, *Damping of Materials and Members in Structural Mechanics*, Pergamon Press Ltd., 4 & 5 Fitzroy Square, London W1, 1968, p. 36.



(a) Lumped-parameter system (b) Isolation to minimize $x_2(t)$ (c) Isolation to minimize F_T

FIG. 3-42. Model of a one-degree-of-freedom system.

discussed in Examples 14 and 15, or (2) an energy dissipating mechanism, which is the present topic of discussion.

Consider the one-degree-of-freedom system in Fig: 3-42(*a*). Damping occurs in the damping mechanism DM, which can be viscous or otherwise. The elements are shown separate in order to form a model for the study. In reality, they may not be separable, such as in an isolator of visco-elastic material. An isolator is shown in Figs. 3-42(*b*) and (c) to minimize the transmission of forces between the madhine m and its foundation.*

Let us find the equivalent damping from energy considerations. Under cyclic strain, the energy dissipation in a damper is measured by the area of the hysteresis loop shown in Fig. 3-43. Without energy dissipation, the cyclic stress strain curve is a line with zero enclosed area, that is, the loop degenerates into a single valued curve. From Fig. 3-42(*a*), if $x_1 = 0$, $x = x_2 - x_1$, and $F(t)$ is sinusoidal, the equation of motion is

$$m\ddot{x} + F_{DM} + kx = F \sin \omega t \tag{3-74}$$

where F_{DM} is the damping force. The energy dissipation per cycle ΔE is

$$\Delta E = \oint F_{DM}\, dx = \int_0^\tau F_{DM}\dot{x}\, dt \tag{3-75}$$

where τ is the period in time per cycle.

*See, for example, J. E. Ruzicka and T. F. Derkley, ***Influence of Damping in Vibration Isolation, SVM-7,*** The Shock and Vibration Information Center, U.S. Department of Defense, 1971.

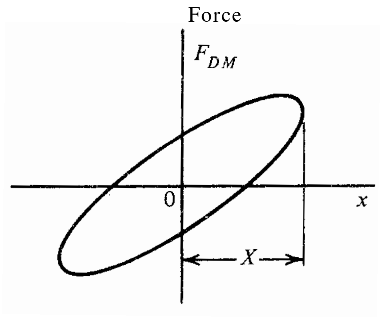


FIG. 3-43. *Hysteresis loop of a damping mechanism.*

If the damping is viscous, the displacement $x(t)$ and the damping force F_{DM} are

$$x = X\sin(\omega t - \phi) \tag{3-76}$$

$$F_{DM} = c\dot{x} = \omega c X\cos(\omega t - \phi) \tag{3-77}$$

where c is the viscous damping coefficient. Combining Eqs. (3-75) to (3-77) gives

$$\Delta E = \int_0^{\tau} (c\dot{x})\dot{x}\,dt = \pi\omega c X^2 \tag{3-78}$$

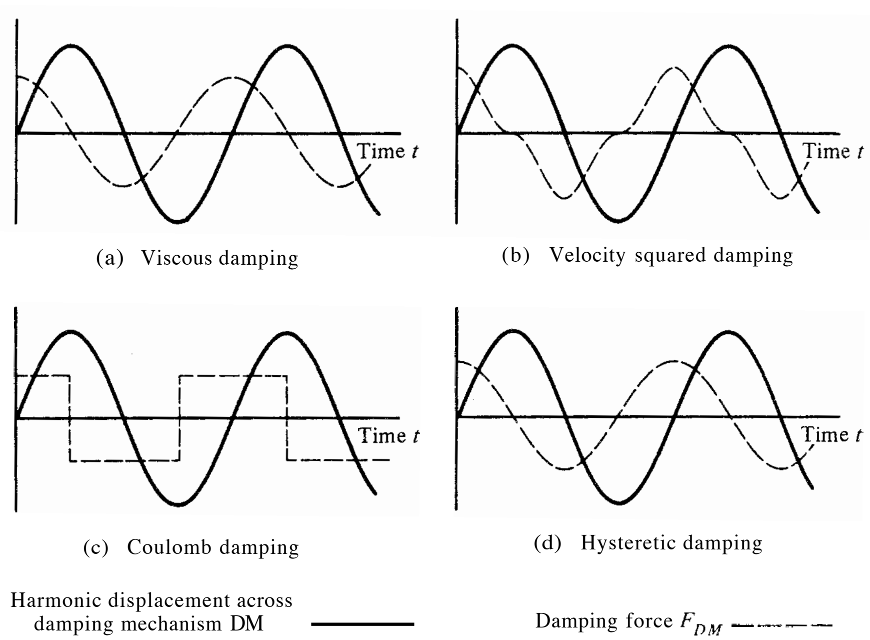


FIG. 3-44. *Wave form of damping force and relative displacement of damping mechanism.*

Combining Eqs. (3-76) and (3-77) yields

$$\left(\frac{x}{X}\right)^2 + \left(\frac{F_{DM}}{\omega c X}\right)^2 = 1 \tag{3-79}$$

Hence the hysteresis loop for viscous damping is an ellipse having the major and minor semiaxis of ωcX and X, respectively.

If the damping is nonviscous, the equivalent viscous damping *coefficient* c_{eq} is obtained from **Eq.** (3-78).

$$\Delta E = \pi\omega c_{eq} X^2 \tag{3-80}$$

Note that the criteria for equivalence are (1)equal energy dissipation per cycle of vibration, and (2) the same harmonic relative displacement. The assumption of harmonic motion is reasonable only for small nonviscous damping. The wave forms of damping forces for harmonic displacements across typical dampers are illustrated in Fig. 3-44.

Example 24. Coulomb Damping

A mass-spring system with Coulomb (dry friction) damping is shown in Fig. 3-45. Assume (1) the-frictional force F_{DM} is proportional to the normal force F_N, and (2) the initial conditions are $x(0) = -x_0$ and $\dot{x}(0) = 0$. Find the motion $x(t)$ and the change in amplitude per cycle.

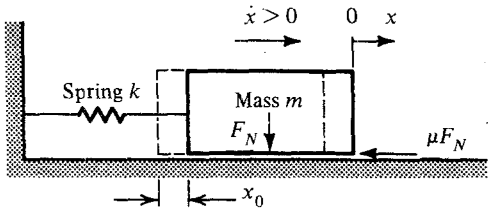


FIG. 3-45. *Coulomb damping.*

Solution:

Let $|F_{DM}| = \mu F_N =$ constant, where μ is a frictional coefficient. If the motion is from left to right ($\dot{x} > 0$), the frictional force is $-\mu F_N$ and vice versa. From **Eq.** (3-74) the equation of motion is

$$m\ddot{x} + \mu F_N \operatorname{sgn}(\dot{x}) + kx = 0$$

where $\operatorname{sgn}(\dot{x}) = \dot{x}/|\dot{x}|$ accounts for the sign change. The corresponding solution is

$$x = A_1 \cos \omega_n t + A_2 \sin \omega_n t - \mu F_N \operatorname{sgn}(\dot{x})/k$$

where $\omega_n^2 = k/m$ and A_1 and A_2 are constants. Substituting the initial conditions, we have

$$x = [\mu F_N \operatorname{sgn}(\dot{x})/k - x_0] \cos \omega_n t - \mu F_N \operatorname{sgn}(\dot{x})/k$$

For the first half cycle ($\omega_n t = \pi$), the displacement is

$$x = x_0 - 2\mu F_N \,\mathrm{sgn}(\dot{x})/k$$

Hence the decrease in amplitude is $2\mu F_N/k$ per half cycle. The amplitude change per cycle for free vibration with Coulomb friction is $4\mu F_N/k$.

Example 25. Equivalent Damping Coefficient

If a force $F \sin \omega t$ is applied to the mass m in Fig. 3-45, find the equivalent viscous-damping coefficient c_{eq} and the magnification factor R.

Solution:

Since the damping force $F_{DM} = \mu F_N$ is constant and the total displacement per cycle is $4X$, the energy dissipation AE per cycle is

$$AE = 4\mu F_N X$$

Comparing this with Eq. (3-80), we have

$$c_{eq} = 4\mu F_N/(\pi \omega X)$$

From Eq. (2-39), the amplitude X of the steady-state response is

$$X = \frac{F}{\sqrt{(k - \omega^2 m)^2 + (\omega c_{eq})^2}} \tag{3-81}$$

This is an implicit equation, since c_{eq} is a function of $\dot{\omega}$ and X. Substituting the expression for c_{eq} and simplifying, the magnification factor R is

$$R = \frac{X}{F/k} = \sqrt{\frac{1 - (4\mu F_N/\pi F)^2}{(1 - r^2)^2}}$$

where $r = \omega/\omega_n$. Since X is real, the equation is valid only if $4\mu F_N/\pi F < 1$. Note that the amplitude at resonance is always theoretically infinite. The resonance amplitude can also be viewed from energy considerations. The energy input per cycle* is $\pi F X$ and the dissipation is $4\mu F_N X$. If $4\mu F_N X/\pi F X < 1$, the excess energy is used to build up the amplitude of oscillation.

Example 26. Quadratic Damping

Quadratic or velocity squared damping is encountered in the turbulent flow of a fluid. Determine the equivalent viscous-damping coefficient and the amplitude of the steady-state response. Assume $x = X \sin wt$.

* Let the force $= F \sin \omega t$ and the harmonic response be $x = X \sin(\omega t - 4)$. At resonance $\phi = 90°$. The energy input per cycle is

$$\oint (\text{force})\, dx = \int_0^{\tau} (F \sin \omega t)\dot{x}\, dt$$

where τ = period. This is integrated to give $\pi F X$.

Solution:

Assume $|F_{DM}| = c_2\dot{x}^2$ where c, is constant and $\dot{x}$ is the relative velocity across the damper. Since damping always opposes the motion, the equation of motion from Eq. (3-74) is

$$m\ddot{x} + c_2 x^2 \operatorname{sgn}(\dot{x}) + kx = F\sin\omega t$$

where $\operatorname{sgn}(\dot{x}) = \dot{x}/|\dot{x}|$ accounts for the sign change.

The energy dissipation per cycle from Eq. (3-75) is

$$\Delta E = 2\int_{-X}^{X} c_2\dot{x}^2\,dx = 2X^3\int_{-\pi/2}^{\pi/2} c_2\omega^2\cos^3\omega t\,d(\omega t)$$
$$= \tfrac{8}{3}\omega^2 c_2 X^3$$

Substituting the ΔE in Eq. (3-80) gives

$$c_{eq} = \frac{8}{3\pi}\,\omega c_2 X$$

Again, c_{eq} is not a constant as assumed for viscous damping.

The amplitude X for the steady-state response is obtained by substituting c_{eq} into Eq. (3-81). Simplifying the resultant equation yields

$$X = \frac{3\pi m}{8c_2r^2}\sqrt{-\frac{(1-r^2)^2}{2} + \sqrt{\frac{(1-r^2)^4}{4} + \left(\frac{8c_2r^2F}{3nkm}\right)^2}}$$

*Example 27. Velocity-nth Power **Damping***

Determine the c_{eq} for the velocity–nth power or exponential damping.

Solution:

The energy dissipation AE per cycle is

$$\Delta E = 2\int_{-X}^{X}(c_n\dot{x}^n)\,dx = 2\omega^n c_n X^{n+1}\int_{-\pi/2}^{\pi/2}\cos^{n+1}\omega t\,d(\omega t)$$

Substituting the AE into Eq. (3-80), we obtain

$$c_{eq} = \omega^{n-1}c_n X^{n-1}\Phi_n$$

where

$$\Phi_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\cos^{n+1}\omega t\,d(\omega t)$$

Example 28. Hysteretic Damping

Assume the damping force in a one-degree-of-freedom system is in phase with the relative velocity but is proportional to the relative displacement across the damper. Find the harmonic response of this system and its equivalent viscous-damping coefficient.

Solution:

Writing the equation of motion in the vectorial form as shown in Sec. *3-5,* we have

$$m\ddot{x} + jhx + kx = Fe^{j\omega t}$$
$$mx + k(1 + j\eta)x = Fe^{j\omega t} \tag{3-82}$$

where *(jhx)* is the damping force, h a constant, $\eta = h/k$ is called the ***loss coefficient,*** and $k(1+j\eta)$ the ***complex stiffness.*** Solving Eq. *(3-82)* by the impedance method, the amplitude X of the harmonic response is

$$X = \frac{F}{|k - \omega^2 m + j\eta k|} = \frac{F/k}{\sqrt{(1-r^2)^2 + \eta^2}} \tag{3-83}$$

where $r = \omega/\omega_n$. Comparing Eqs. *(3-81)* and (3-83), the equivalent viscous-damping coefficient c_{eq} is

$$c_{eq} = h/\omega \tag{3-84}$$

Solid damping, hysteretic damping, and structural damping are terms commonly used to describe the internal damping of material. It is assumed that the energy dissipation per cycle is independent of frequency and is proportional to the square of the strain amplitude. Substituting Eq. (3-84) into (3-80), we obtain $\Delta E = h\pi X^2$, which confirms the assumption. The nature of structural damping is rather complex. For mild steel, the energy dissipation is proportional to $X^{2.3}$. For other cases, the value of the amplitude exponent may range from 2 to 3. The damping of a material may decrease slightly with increasing frequency instead of being constant. In contrast, the common viscous-damping theory assumes that the loss coefficient increases linearly with frequency.

Owing to their high damping characteristics, visco-elastic materials have gained importance in vibration control in recent years. The physical properties of such materials are more complex than those of metals. The properties, are influenced by the operating conditions, the past history, and the geometry of the damping mechanism. The variation of properties with temperature and frequency of a typical visco-elastic material is shown in Fig. 3-46.* The complex modulus $E(1+j\eta)$ is defined by the relation

$$(\text{Stress}) = E(1 + j\eta)(\text{strain})$$

where η is the loss coefficient. Note that high damping can be achieved in the transitional region. Furthermore, if the spring constant of a damping mechanism increases with preloading, it is feasible to tune a dynamic absorber by adjusting its preload.

* D. J. Jones, ***Material Damping,*** ASA Damping Conference, Cleveland, Ohio, Nov. 21, **1968.**

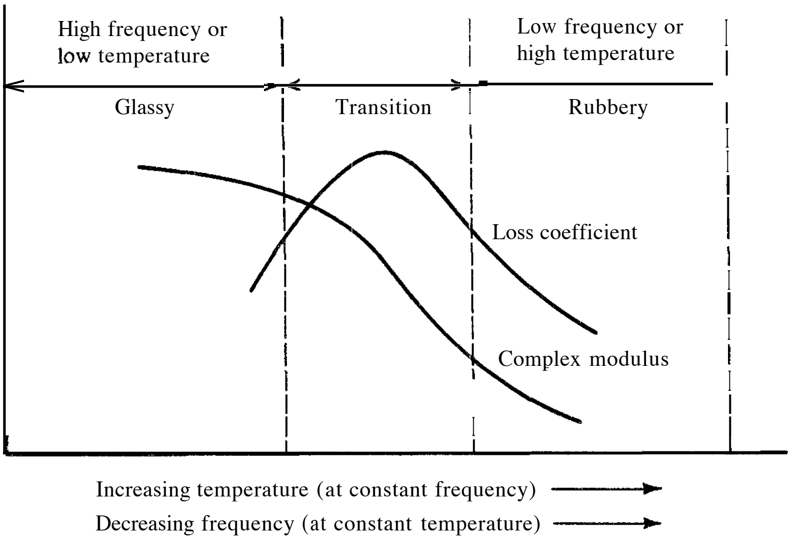


FIG. 3-46. *Temperature and frequency effect of a "typical" polymer.*

The frequency dependence of the complex modulus is often considered as a disadvantage in using visco-elastic material for isolation mounting. However, for a tuned dynamic absorber with damping (see Sec. 4-8), the frequency dependency also moves the "resonant frequency" of the absorber progressively higher or lower with the exciting frequency. Thus, the absorber is effective for a greater frequency range of operation.*

3-9 SUMMARY

The theory of one-degree-of-freedom systems is applied to a variety of problems, the model of which is as shown in Fig. 2-6. The generalized equation of motion is

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F_{eq}(t) \tag{3-1}$$

The examples in the chapter are grouped to illustrate the equivalent quantities m_{eq}, c_{eq}, k_{eq}, and $F_{eq}(t)$. The emphasis is on problem formulation and interpretation, since the general theory was developed in the last chapter.

The equivalent mass m_{eq} and equivalent spring k_{eq} are illustrated in Examples 1 to 8.

* J. C. Snowdon, *Vibration and Shock in Damped Mechanical Systems,* John Wiley & Sons, Inc., New York, 1968, p. 96.

Damped free vibration is examined in Sec. 3-3. The logarithmic decrement in Eq. (3-12) is a convenient way to measure the damping in a system. As shown in Fig. 3-10, it takes relatively few cycles for the transient vibration to die out, even for lightly damped systems.

Examples of F_{eq} in vibration testing and control for undamped systems are shown in Sec. 3-4. A lightly damped system can be considered as undamped, except at near resonance. The harmonic response of an undamped system is theoretically infinite at resonance, Eq. (3-16). It takes time, however, for the amplitude to build up at resonance, Eq. (3-18). In practice, if a resonance frequency is passed through quickly, a machine may operate at a desired speed between resonant frequencies.

Systems with damping under harmonic excitation are treated in Sec. 3-5 in six cases. The rotating unbalance in machines in Case 1 and critical speed of shafts in Case 2 are two views of the same problem, because the excitation in both are due to an unbalance in the machine. Hence the response curves in Fig. 3-16 can be used to present both cases.

Vibration isolation and transmissibility in Case 3 and the response to moving support in Case 4 are again two aspects of the same problem as evident by comparing Eqs. (3-32) and (3-37). The response of both cases are shown in Fig. 3-25.

The seismic instrument in Case 5 uses the relative displacement $x(t)$ between a mass and its base to measure the motion $x_1(t)$ of the base. The equation of motion in Eq. (3-39) is of the same form as those for Cases 1 and 2. Hence the response can also be represented in Fig. 3-16, although they are distinct types of problems. Two types of instruments can be constructed from this general theory, depending on the excitation frequency ω and the natural frequency ω_n of the system. If $\omega \gg \omega_n$, we have a vibrometer for displacement measurement and $x(t) = -x_1(t)$, regardless of the damping as shown in Fig. 3-16. If $\omega \ll \omega_n$, we have an accelerometer and $x(t)$ is proportional to $\ddot{x}_1(t)$. Appropriate damping is necessary in an accelerometer in order to minimize the amplitude and phase distortions.

A periodic excitation can be expressed as a Fourier series. The system response due to each harmonic component of the periodic excitation can be calculated as illustrated in Fig. 3-37. The total response is obtained by superposition as shown in Eq. (3-65). Hence this is a generalization of the harmonic excitation in Sec. 3-5.

Shock is a transient phenomenon. The shock spectrum in Sec. 3-7 describes the effect of shock on the assumption that shock excitations having the same spectrum would produce similar effect on the system,

Damping is seldom purely viscous in a real system. For nonviscous damping, an equivalent viscous damping coefficient c_{eq} is defined in Eq. (3-80). The criteria for equivalence are (1) equal energy dissipation per cycle of vibration, and (2) harmonic oscillations of the same amplitude.

PROBLEMS

Assume all the systems in the figures to follow are shown in their static equilibrium positions.

3-1 Find the natural frequency of the system in Fig. **P3-1(*a*).** Assume that **(1)** the cantilevers are of negligible mass and **(2)** their equivalent spring constants are k, and k_3.

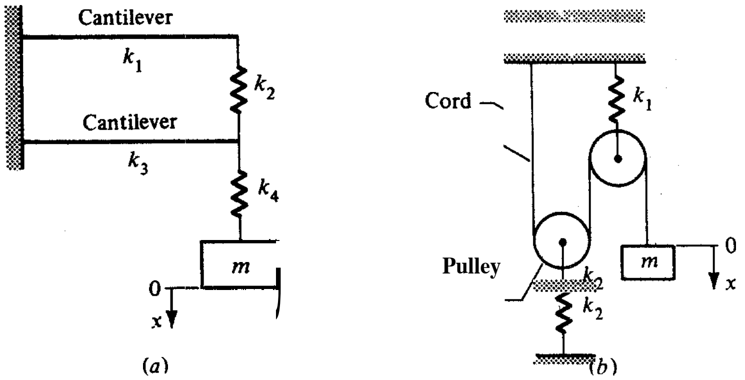


FIG. **P3-1.**

3-2 Neglecting the mass of the pulleys, find the natural frequency of the system in Fig. **P3-1(*b*).**

3-3 A mass m is attached to a rigid bar of negligible mass as shown in Fig. **P3-2(*a*).** Find the natural frequency of the system, if **(a)** the bar is constrained to remain horizontal while m oscillates,vertically; (b) the bar is free to pivot at the hinges A and B. (c) Show that the natural frequency determined in part **a** is higher than that of part **b.**

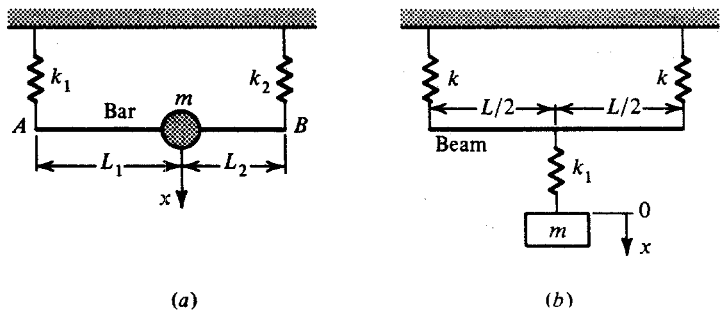


FIG. P3-2.

3-4 A mass m is suspended as shown in Fig. P3-2(*b*). If the beam is of negligible mass and its deflection δ is given by the equation $\delta = PL^3/48EI$, find the natural frequency of the system.

3-5 Referring to Fig. **2-1(*a*),** let k = 7 **kN/m** and m = **18** kg. **(a)** Find the natural frequency of the system. **(b)** If the ends **of** the spring are fixed and the mass m is attached to the midpoint of the spring, find the natural frequency. (c) If the ends of the spring are fixed and m is attached to some intermediate point of the spring, show that the natural frequency of this configuration is higher than that of part b.

3-6 A rocker arm assembly is shown in Fig. P3-3(a). Let m_A =mass of rocker arm and J_A =the mass moment of inertia about the pivot A. Find the equivalent mass m_{eq} and the equivalent spring k_{eq} of the system referring to the x coordinate.

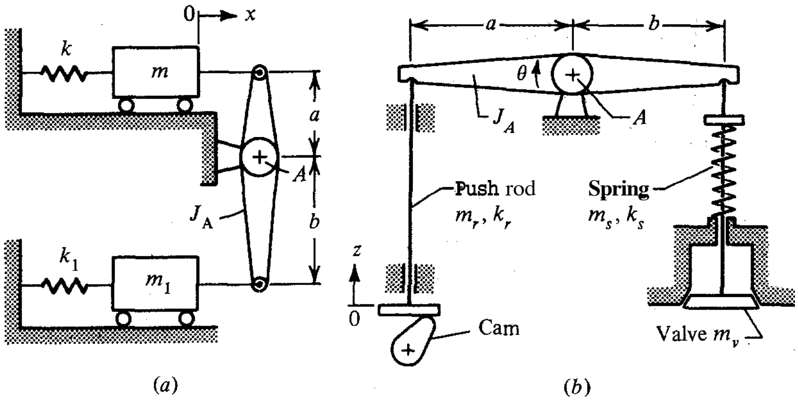


FIG. P3-3.

3-7 An engine valve arrangement is shown in Fig. P3-3(b), where J_A is the mass moment of inertia of the rocker arm about the pivot A. Assume the effective mass m_r and the effective stiffness k_r of the pushrod are known. Reduce the valve arrangement to an equivalent mass-spring system.

3-8 A mechanism is shown schematically in Fig. P3-4(a). Assuming that the tension of the spring k_3 is constant, derive the equation of motion of the system.

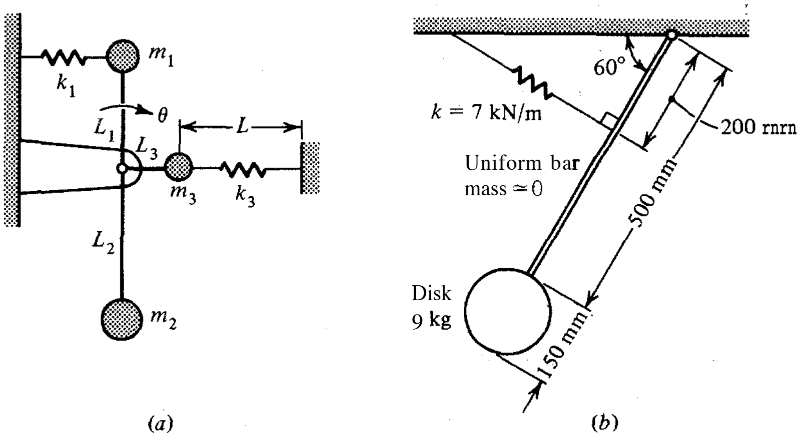
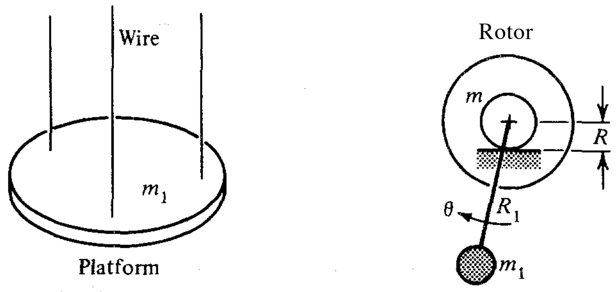


FIG. P3-4.

3-9 A machine component is depicted as a pendulum shown in Fig. P3-4(*b*). Determine its natural frequency by: (a) Newton's second law; (b) the energy method.

3-10 The mass moment of inertia J_{cg} of a connecting rod of mass m is determined by placing the rod on a horizontal platform of mass m, and timing the periods of oscillation. The platform, shown in Fig. P3-5(*a*), is suspended by equally spaced wires. With the platform empty and an amplitude of 6°, the period is τ_1. With the mass center of the rod coinciding with that of the platform and an amplitude of 8°, the period is τ_2. Find J_{cg} of the connecting rod.

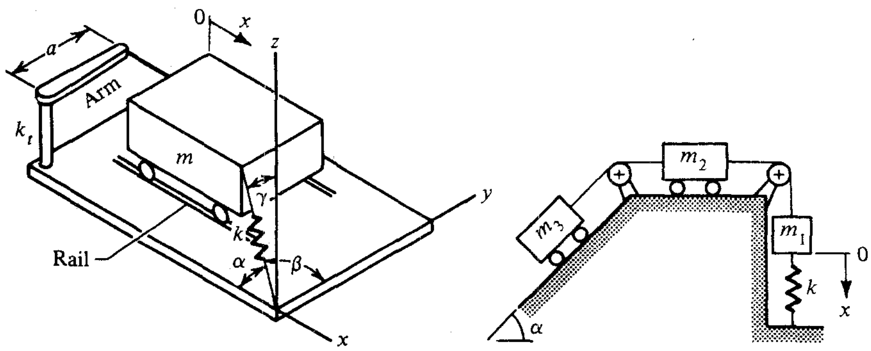


(*a*) (*b*)

Fig. P3-5.

3-11 The mass moment of inertia J_0 of the rotor of an electrical generator of mass m is found by attaching a small mass m_1 at a distance R_1 from its longitudinal axis and timing the periods of oscillation. The test setup is shown in Fig. P3-5(*b*). (a) Find J_0 of the rotor. (b) Show that small variation of R will have the least effect when $R_1 = (m/m_1 + 1)R$.

3-12 A machine component is shown in Fig. P3-6(*a*). The mass m is constrained by rails to move only in the x direction. Neglecting the mass of the arm, find the equation of motion.



(*a*) (*b*)

Fig. P3-6.

3-13 A machine part is schematically shown in Fig. **P3-6(*b*).** Find the equation of motion.

3-14 Find the equation of motion of the mass m for the system shown in Fig. **P3-7(*a*).** Assume that the horizontal bar is rigid and is of negligible mass.

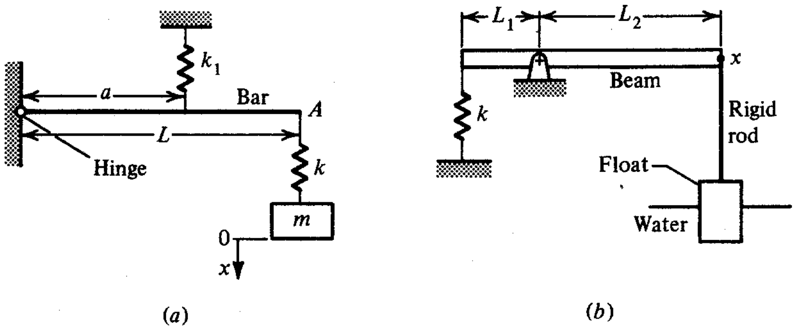


FIG. P3-7.

3-15 A walking-beam configuration, consisting of a uniform beam of mass m_1 and a cylindrical float of cross-sectional area A, is shown in Fig. **P3-7(*b*).** If the mass of the float and the rod is m_2, determine the equation of motion.

3-16 For a one-degree-of-freedom system, if, m = 7 kg, k = 6 **kN/m,** and c = 35 N **. s/m,** find: (a) the damping factor ζ, (b) the logarithmic decrement and (c) the ratio of any two consecutive amplitudes.

3-17 From the data in Prob. 2-14, find the logarithmic decrement for each of the given sets of initial conditions.

3-18 A device bought from a surplus store is depicted as a **one-degree-of-**freedom system. It is desired to find: (a) its natural frequency, (b) the mass moment of inertia of the rotor, and (c) the damping required for it to be critically damped. It is not possible, however, to disassemble the device. It is found that (1) when the rotor is turned **30°,** a **torque** of 0.35 N **.** m is needed to maintain this position, (2) when the rotor is held in this position and released, it swings to –5.5' and then to **1°,** and (3) the time of the swing is 1.0 s. Calculate the required information.

3-19 Derive the equations of motion for the systems in Fig. P3-8. Assume the bars are rigid and of negligible mass.

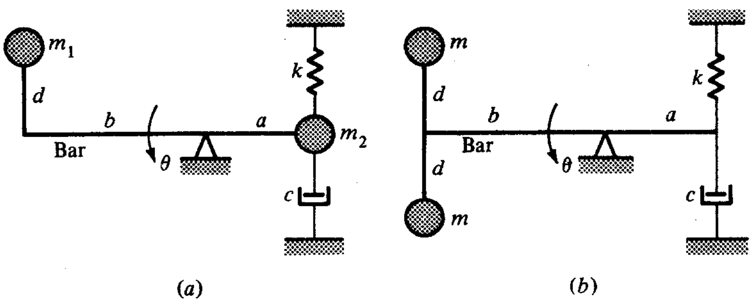
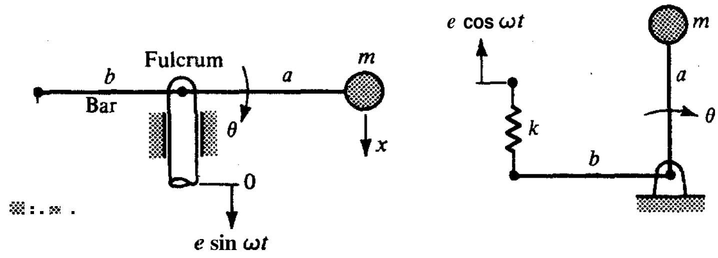


FIG. P3-8.

3-20 Derive the equations of motion for the system shown in Fig. P3-7(*a*) if an excitation force $F \sin \omega t$ is applied to: (a) the mass m (b) the free end A of the bar.

3-21 A force $F \sin \omega t$ is applied to the mass m of the system shown in Fig. 2-1(*a*). If $\omega = (1+\varepsilon)\omega_n$, determine the motion of m. Assume zero initial conditions and $\varepsilon \ll 1$.

3-22 A harmonic motion is applied to each of the systems shown in Fig. P3-9. Derive the equations of motion.



(*a*) (*b*)

FIG. P3-9.

3-23 Referring to Fig. P3-9(*a*), the general position of the fulcrum can be above or below the static equilibrium position of the system and the bar can be rotated clockwise or counterclockwise. Show that the equation of motion derived in Prob. 3-22 is true for all positions of the system.

3-24 A wide-flange I beam is cantilevered from the foundation of a building. The beam is 2 m in length with a total mass of 60 kg. The I of the beam section is 30×10^6 mm^4. A construction worker places a small electric motor of 4 kg at the end of the beam. The mass of the armature of the motor is 1.5 kg with an eccentricity of 0.05 mm. If the motor speed is 3,600 rpm, estimate the amplitude of vibration at the end of the beam.

3-25 Show analytically that the maximum value of the curves in Fig. 2-8 occur at $r<1$. Sketch a locus through the maxima of the curves.

3-26 Show analytically that the maximum values of the curves in Fig. 3-16 occur at $r>1$. Sketch a locus through the maxima of the curves.

3-27 A table for sorting seeds requires a reciprocating motion with a stroke of 1.0 mm and frequency from 2 to 20 Hz. The excitation is provided by an eccentric weight shaker. The total mass of the table and shaker is 200 kg. (a) Propose a scheme for mounting the table. (b) Specify the spring constant, the damping coefficient, and the unbalance of the exciter.

3-28 A machine of 100 kg mass has a 20 kg rotor with 0.5 mm eccentricity. The mounting springs have $k = 85$ kN/m and the damping is negligible. The operating speed is 600 rpm and the unit is constrained to move vertically. (a) Determine the dynamic amplitude of the machine. (b) Redesign the mounting so that the dynamic amplitude is reduced to one half of the original value, but maintaining the same natural frequency.

3-29 A variable-speed counter-rotating eccentric-weight exciter is attached to a machine to determine its natural frequency. With the exciter at 1,000 rpm, a stroboscope shows that the eccentric weights of the exciter are at the top the instant the machine is moving upward through its static equilibrium position. The amplitude of the displacement is 12 mm. The mass of the machine is 500 kg and that of the exciter is 20 kg with an unbalance of 0.1 kg · m. Find (a) the natural frequency of the machine and its mounting and (b) the damping factor of the system.

3-30 A rotating machine for research has an annular clearance of 0.8 mm between the rotor and the stator. The mass of the rotor is 36 kg with an unbalance of 3×10^{-3} kg . m. The rotor is mounted symmetrically on a round shaft, 300 mm in length and supported by two bearings. The operating speed ranges from 600 to 6,000 rpm. If the dynamic deflection of the shaft is to be less than 0.1 mm, specify the size of the shaft.

3-31 A circular disk of 18 kg is mounted symmetrically on a shaft, 0.75 m in length and 20 mm in diameter. The mass center of the disk is 3 mm from its geometric center. The unit is rotated at 1,000 rpm and the damping factor ζ is estimated to be 0.05. (a) Compare the static stress of the shaft with the dynamic stress at the operating speed. (b) Repeat part a for a shaft of 30 mm in diameter.

3-32 It is proposed to use a three-cylinder two-stroke-cycle diesel engine to drive an electric generator at 600 rpm. The generator consists of a 2×10^3 kg rotor mounted on a hollow shaft, 2 m in length with a 200-mm OD and a 100-mm bore. A preliminary test shows that, when the rotor is suspended horizontally with its axis 0.95 m from the point of suspension, the period of oscillation is $2\pi/3$ s. If you are the consulting engineer, would you approve this proposal?

3-33 A turbine at 6,000 rpm is mounted as shown in Fig. P3-10(*a*). The 14 kg rotor has an unbalance of 2.8×10^{-3} kg · m. (a) Neglecting the mass of the shaft, find the amplitude of vibration and the force at each bearing for a 20-mm diameter shaft. (b) Repeat for a 25-mm shaft. (c) Repeat for a 30-mm shaft. (d) Estimate the error due to neglecting the mass of the shaft.

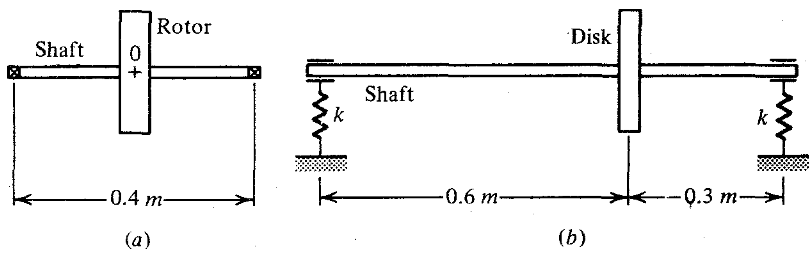


FIG. P3-10.

3-34 A 180-kg steel disk is mounted on a 100-mm OD and 75-mm ID shaft as shown in Fig. P3-10(*b*). (a) Neglecting the flexibility of the bearing supports, find the critical speed of the assembly. (b) If the bearings are flexible with a spring constant $k = 70 \times 10^6$ N/m in any direction normal to the shaft axis, find the change in critical speed. Assume that the mass of the shaft and the gyroscopic effect of the disk are negligible.

3-35 A 140-kg disk is mounted on a 75-mm diameter shaft as shown in Fig. P3-10(*b*). The bearing supports are essentially rigid in the vertical direction but are flexible with a spring constant $k = 50 \times 10^6$ N/m each in the horizontal direction. Find the critical speeds of the assembly.

3-36 Show analytically that the crossover points of the transmissibility curves in Figs. 3-25 and 3-26 occur at $r = \sqrt{2}$.

3-37 A refrigeration unit of 30-kg mass operates at 700 rpm. The unit is supported by three equal springs. (a) Specify the springs if 10 percent or less of the unbalance is transmitted to the foundation. (b) Verify the calculation, using Fig. 3-27.

3-38 A vertical single-cylinder diesel engine of 500-kg mass is mounted on springs with $k = 200$ kN/m and dampers with $\zeta = 0.2$. The rotating parts are well-balanced. The mass of the equivalent reciprocating parts is 10 kg and the stroke is 200 mm. Find the dynamic amplitude of the vertical motion, the transmissibility, and the force transmitted to the foundation, if the engine is operated at (a) 200 rpm; (b) at 600 rpm.

3-39 A 50-kg rotor is mounted as shown in Fig. P3-11(*a*). It has an unbalance of 0.06 kg · m and operates at 800 rpm. If the dynamic amplitude of the rotor is to be less than 6 mm and it is desired to have low transmissibility, specify the springs and the dampers for the mounting.

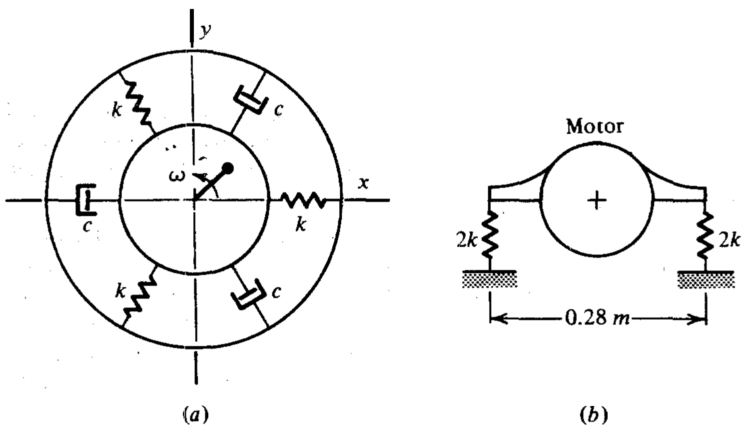


FIG. P3-11.

3-40 A 15-kg electric motor is supported by four equal springs as shown in Fig. P3-11(*b*). The stiffness of each spring is 2.5 kN/m. The radius of gyration of the motor assembly about its shaft axis is 100 mm. The operating speed is 1,800 rpm. Find the transmissibility for the vertical and the torsional vibrations.

3-41 An instrument in an aircraft is to be isolated from the engine vibrations, ranging from 1,800 to 3,600 cycles per minute. If the damping is negligible and the instrument is of 20-kg mass, specify the springs for the mounting for 80 percent isolation.

3-42 A 250-kg table for repairing instruments is isolated from the floor by springs with $k = 20$ kN/m and dampers with $c = 4$ kN . s/m. If the floor vibrates vertically ±2.5 mm at a frequency of 10 Hz, find the motion of the table.

3-43 Referring to the vehicle suspension problem in Example 19, if $X_1 = 0.1$ m, find the amplitude X_2 when the speed of the trailer is: (a) 70 km/hr; (b) 120 km/hr.

3-44 A body m, mounted as shown in Fig. P3-12(*a*), is dropped on a floor. Assume that, when the base first contacts the floor, the spring is unstressed and the body has dropped through a height of 1.5 m. Find the acceleration $\ddot{x}(t)$ of m. If m = 18 kg, c = 72 N . s/m, and k = 1.8 kN/m, determine the maximum acceleration of m.

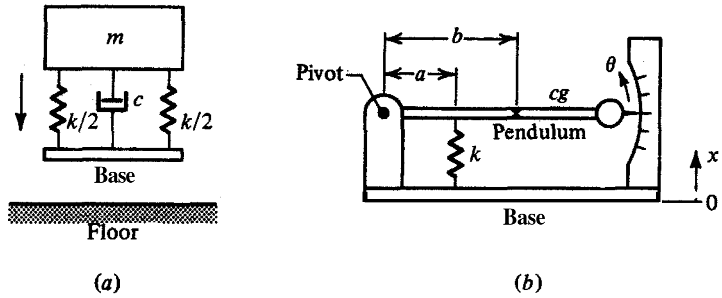


FIG. P3-12.

3-45 A vibrometer for measuring the rectilinear motion $x(t)$ is shown in Fig. P3-12(*b*). The pivot constrains the pendulum to oscillate in the plane of the paper and viscous damping exists at the pivot. Derive the equation of motion of the system.

3-46 A torsiograph is a seismic instrument to measure the speed fluctuation ot a rotating shaft. A torsiograph consisting of a hollow cylinder of 0.5 kg with a 40-mm radius of gyration is mounted coaxially with the shaft and connecting to it by a spiral spring. Assuming that (1) viscous damping exists between the cylinder and the shaft, (2) the average shaft speed is 600 rpm, and (3) the frequency of fluctuations varies from 4 to 8 times the shaft speed, specify the spring constant and the damping coefficient if the torsiograph is to measure relative displacement.

3-47 A vibrometer to measure the vibrations of a variable speed engine is schematically shown in Fig. 3-29. The vibrations consist of a fundamental and a second harmonic. The operating speed ranges from 500 to 1500 rpm. It is desired to have the amplitude distortion less than 4 percent. Determine the natural frequency of the vibrometer if: (a) the damping is negligible; (b) the damping factor $\zeta = 0.6$.

3-48 An accelerometer with $\zeta = 0.6$ is used to measure the vibrations described in Prob. 4-47. The amplitude distortion is to be less than 4 percent. (a) From Fig. 3-32, find the natural frequency of the accelerometer. (b) If the engine operates at 1,000 rpm, find the amplitude distortion of the second harmonic. (c) Find the phase distortion from Fig. 3-34 and calculate the phase shift in unit of time.

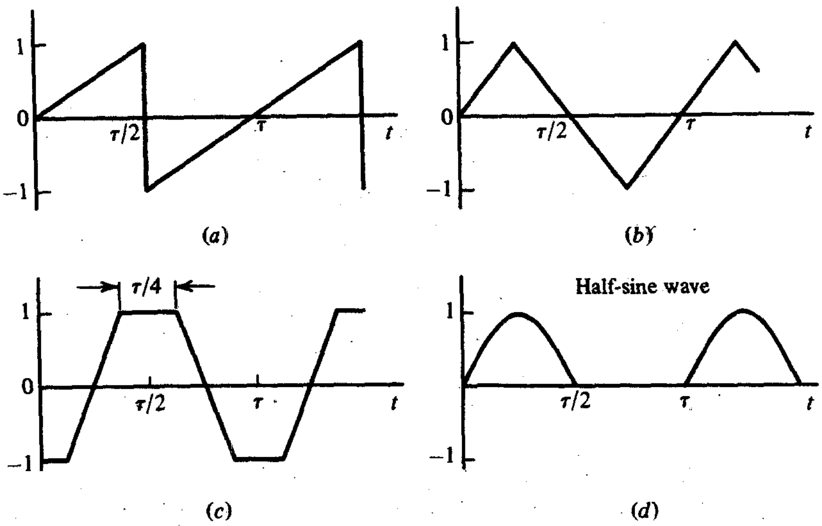


FIG. P3-13.

3-49 Find the Fourier series expansions of each of the periodic functions shown in Fig. P3-13.

3-50 From the answers in **Prob.** 3-49, find the Fourier spectrum of **the** periodic functions in Figs. **P3-13(*a*)** and (b).

3-51 Find the Fourier series expansions of the periodic functions in Fig. P3-13 if each of the functions is delayed by an amount $\tau/4$.

3-52 From the answers in Prob. 3-51, find the Fourier spectrum of the periodic functions of parts a and **b.**

3-53 The excitation of a system has **two** harmonic components.

$$3\ddot{x} + 18\dot{x} + 675x = 6 \sin 10t + 3 \sin(20t + 3\pi)$$

(a) Sketch the wave form of the excitation. (**b**) **Use** the impedance method in Eq. (3-62) and (3-63) to find the steady-state response due to each of the **harmonic** components. (c) Sketch the wave form of the composite **steady-**state response.

3-54 If the system in Fig. 3-38 is actuated by a cam with the profile as shown in Fig. **P3-13(c),** find the steady-state response of the system. Assume that $m = 170$ kg, $k_1 = k = 7$ **kN/m,** $c = 1.7$ **kN** · s/m, total cam lift = 50 mm, and the cam speed = 60 **rpm.**

3-55 Derive the transmissibility equation for each of the systems shown in Fig. P2-5. (See Prob. 2-24, Chap. 2).

3-56 A periodic force, with the **waveform** as shown in Fig. **P3-13(*b*),** is applied to a mass-spring system. Will there be a resonance if the fundamental frequency of the excitation is one-half of the natural frequency of the **system?**

3-57 Find the relative motion $x = (x_2 - x_1)$ of the mass m in Fig. 3-23, if the base is given an excitation: (a) $\ddot{x}_1 = g$; (**b**) $\dot{x}_1 = Ce^{-\alpha t}$, where C and a are constants. Assume that the damping is negligible and the system is 'initially at rest.

3-58 For the system with Coulomb damping shown in Fig. 3-45, deduce from energy considerations that the amplitude decay for the free vibration is $4F/k$ per cycle, where F is the frictional force.

3-59 For the system with Coulomb friction shown in Fig. 3-45, assume that $m = 9$ kg, $k = 7$ kN/m, and the friction coefficient $\mu = 0.15$. If the initial conditions are $x(0) = 25$ mm and $\dot{x}(0) = 0$, find: (a) the dectease in displacement amplitude per cycle, (b) the maximum velocity, (c) the decrease in velocity amplitude per cycle, and (d) the position at which the body m would stop.

3-60 For the system with Coulomb damping in Fig. 3-45, let an excitation $F_0 \sin \omega t$ be applied to the mass m. (a) Use Eq. (3-75) to show that the energy dissipation per cycle is $4FX$, where F is the frictional force. (b) Show that the transmissibility TR is infinite at resonance. (c) Find TR for the frequency ratios $r \gtrless \sqrt{2}$, where $r = \omega/\omega_n$.

3-61 For a one-degree-of-freedom system with velocity-squared damping, assume the force applied to the mass is $F_0 \sin wt$. (a) Find the resonance amplitude from energy considerations. (b) Check part a from the expression for X in Example 26.

3-62 A machine of 350-kg mass and 1.8-kg. m eccentricity is mounted on springs and a damper with velocity squared damping. The damper consists of a 70-mm diameter cylinder-piston arrangement. The piston has a nozzle for the passage of the damping fluid, the density of which is $\rho = 960$ kg/m^3. The natural frequency of the system is 5 Hz. Assuming that the equivalent viscous-damping factor $\zeta_{eq} = 0\ 2$ at resonance, determine: (a) the resonance amplitude; (b) the diameter of the nozzle if the pressure drop across the nozzle is $p = (\rho/2)(\text{velocity})^2$, where (velocity) is that at the throat of the nozzle.

Computer problems:

3-63 Use the program **TRESP1** listed in Fig. 9-1(a) to find the free vibration of the system $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$. Choose appropriate values for ζ, ω_n, $x(0)$, and $\dot{x}(0)$. Select about two cycles for the duration of the run.

3-64 (a) Repeat Prob. 3-63 by modifying the program for plotting, as illustrated in Fig. 9-4(a). (b) Plot the results using the program **PLOTFILE** listed in Fig. 9-5(a).

3-65 Consider the equation of motion and the harmonic response

$$m\ddot{x} + c\dot{x} + kx = F_{eq} \sin \omega t$$

$$x = X \sin(\omega t - \phi)$$

The response can be expressed as shown in Eqs. (3-25) and (3-26). (a) Modify the program **FRESP1**, listed in Fig. 9-6(a), to write the values of the amplitude ratio $X/(F_{eq}/k)$ and the phase angle in separate files for plotting. Let the damping factor $\zeta = 0.2$ and $0.3 \le r \le 4.5$. (b) Plot the results using the program **PLOTFILE** listed in Fig. 9-5(a).

3-66 Repeat Prob. 3-65 but plot for a set of values of ζ, as illustrated in Figs. 2-10 and 2-11. Let $\zeta = 0.1$, 0.2, 0.3, 0.4, 0.55, 0.7, and 1.0.

3-67 (a) Classify and tabulate the harmonic response equations for Cases 1 to 5 in Sec. 3-5. (b) Write a program to plot the amplitude-ratio versus frequency-ratio r for each type of response for a range of damping factor ζ. In other words, the object is to plot the response curves illustrated in Figs. 3-16, 3-25, and 3-26. Hint: Modify the program **FRESP1,** as listed in Fig. 9-6(a), and use PLOTFILE, listed in Fig. 9-5(a).

3-68 Repeat Prob. 2-32, 2-33, 2-34, or 2-35, but (1) modify the program for plotting, as illustrated in Fig. 9-4(a) and (2) write the program such that only every nth data point is plotted. Use $n = 2$ for this program.

4

Systems with More Than One Degree of Freedom

4-1 INTRODUCTION

The degree of freedom of discrete systems was defined in Chap. 2. Since there is no basic difference in concept between systems with two or more degrees of freedom, we shall introduce multi-degree-of-freedom systems from the generalization of systems with two degrees of freedom. Computers, however, are mandatory for the numerical solution of problems with more than two degrees of freedom.

An n-degree-of-freedom system is described by a set of n simultaneous ordinary differential equations of the second-order. The system has as many natural frequencies as the degrees of freedom. A mode of vibration is associated with each natural frequency. Since the equations of motion are coupled, the motion of the masses are the combination of the motions of the individual modes. If the equations are uncoupled by the proper choice of coordinates, each mode can be examined as an independent one-degree-of-freedom system.

To implement the topics enumerated above, computer sub-routines are listed in App. C, such as for (1) finding the characteristic equation from the equations of motion, (2) solving the characteristic equation for the natural frequencies, (3) obtaining the modal matrix in order to uncouple the equations of motion, and (4) solving the corresponding one-degree-of-freedom system. Computer solutions are given as home problems. For purpose of organization, typical computer programs are grouped and illustrated in Chap. 9.

We shall begin by formulating the equations of motion from Newton's second law, and then discuss natural frequencies, coordinate coupling and transformation, modal analysis, and applications. The method of influence coefficients is presented in the latter part of the chapter. Orthogonality of the modes and additional concepts for a better understanding of the material will be discussed in Chap. 6.

4-2 EQUATIONS OF MOTION: NEWTON'S SECOND LAW

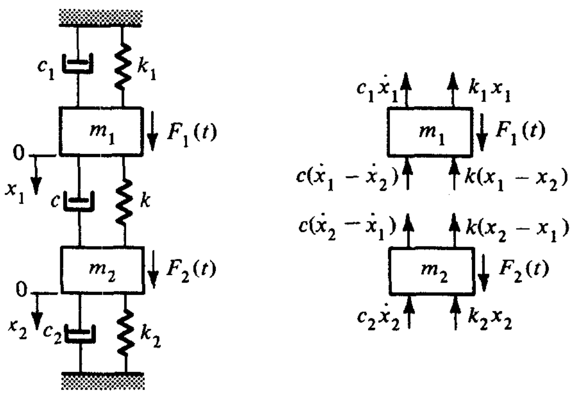
The equations of motion for the two-degree-of-freedom system in Fig. 4-1(a) can be derived by applying Newton's second law to ***each*** of the masses. Assume the damping is viscous and the displacements $x_1(t)$ and $x_2(t)$ measured from the static equilibrium positions of the masses. Summing the ***dynamic forces*** in the vertical direction on each mass shown in the free-body sketches, we get

$$m_1\ddot{x}_1 = -k_1x_1 - k(x_1 - x_2) - c_1\dot{x}_1 - c(\dot{x}_1 - \dot{x}_2) + F_1(t)$$
$$m_2\ddot{x}_2 = -k_2x_2 - k(x_2 - x_1) - c_2\dot{x}_2 - c(\dot{x}_2 - \dot{x}_1) + F_2(t)$$

which can be rearranged to

$$\begin{aligned} m_1\ddot{x}_1 + (c + c_1)\dot{x}_1 + (k + k_1)x_1 - c\dot{x}_2 - kx_2 &= F_1(t) \\ -c\dot{x}_1 - kx_1 + m_2\ddot{x}_2 + (c + c_2)\dot{x}_2 + (k + k_2)x_2 &= F_2(t) \end{aligned} \qquad \textbf{(4-1)}$$

where $F_1(t)$ and $F_2(t)$ are the excitation forces applied to the respective masses. Note that the equations are not independent, because the equation for m_1 contains terms in x_2 and $\dot{x}_2$. Hence the coupling terms in the first equation in Eq. (4-1) are $-(c\dot{x}_2 + kx_2)$. Similarly, the coupling terms in the second equation are $-(c\dot{x}_1 + kx,)$. In other words, the motion $x_1(t)$ of m_1 is influenced by the motion $x_2(t)$ of m_2 and vice versa. Coordinate coupling will be **discussed** in detail in **Sec.** 4-4.



(a) System (b) Free-body sketches

FIG. 4-1. A ***two-degree-of-freedom system.***

For conciseness, Eq. (4-1) can be expressed in matrix notations as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} c+c_1 & -c \\ -c & c+c_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k+k_1 & -k \\ -k & k+k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix} \quad \textbf{(4-2)}$$

or

$$M\{\ddot{x}\}+C\{\dot{x}\}+K\{x\}=\{F(t)\} \quad \textbf{(4-3)}$$

By simple matrix operations, it can be shown that Eqs. (4-1) and (4-2) are equivalent. The quantities in Eq. (4-3) can be identified by comparing with Eq. (4-2). The 2×2 matrices M, C, and K are called the mass matrix, damping matrix, and *stiffness* matrix, respectively. The 2×1 matrix **{x)** is called the displacement vector. The corresponding velocity vector is {x} and the acceleration vector is {f}. The 2×1 matrix $\{F(t)\}$ is the force vector.

It will be shown in **Sec.** 4-4 that if another set of coordinates $\{q_1q_2\}$ is used to describe the motion of the same system, the values of the elements in the matrices M, C, and K will differ from those shown in Eq. (4-2). The inherent properties of the system, such as natural frequencies, must be independent of the coordinates used to describe the system. Hence the general form of the equations of motion of a two-degree-of-freedom system is

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}+\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}+\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}=\begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} \quad \textbf{(4-4)}$$

or

$$M\{\ddot{q}\}+C\{\dot{q}\}+K\{q\}=\{Q(t)\} \quad \textbf{(4-5)}$$

The 2×2 matrices M, C, and K associated with the coordinates {q} can be identified by comparing the last two equations. The 2×1 matrix $\{Q(t)\}$ is the force vector associated with the displacement vector {q}.

Generalizing the concept, Eq. (4-5) also describes the motion of an n-degree-of-freedom system if the matrices M, C, and K are of nth-order, that is,

$$M=[m_{ij}], \qquad C=[c_{ij}], \qquad K=[k_{ij}] \quad \textbf{(4-6)}$$

where $i, j=1,2,3,\ldots, n$. The coefficients m_{ij}, c_{ij}, and k_{ij} are the elements of the matrices M, C, and K, respectively. The generalized coordinates {q} and the generalized force vector $\{Q(t)\}$ are

$$\{q\}=\{q_1 \cdots q_n\} \quad \textbf{(4-7)}$$

$$\{Q(t)\}=\{Q_1(t)\cdots Q_n(t)\} \quad \textbf{(4-8)}$$

Hence Eq. (4-5) is also the general form of the equations of motion of an n-degree-of-freedom system.

4-3 UNDAMPED FREE VIBRATION: PRINCIPAL MODES

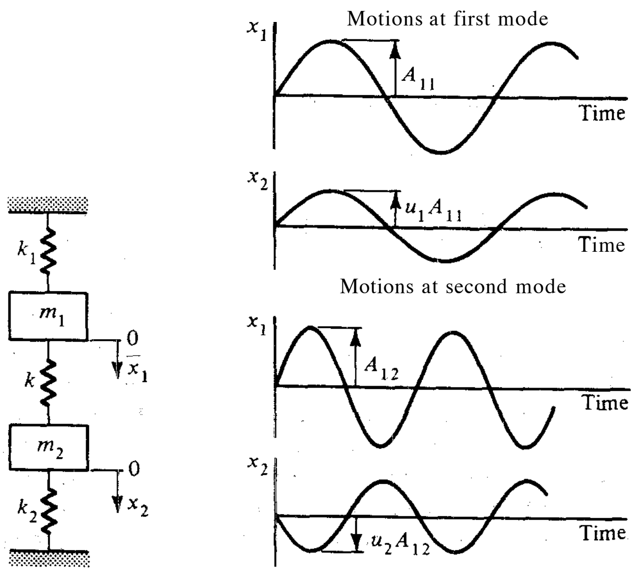
A dynamic system has as many natural frequencies and modes of vibration as the degrees of freedom. The general motion is the superposition of the modes. We shall discuss (1) a method to find the natural frequencies, and (2) the modes of vibration of an undamped system at its natural frequencies.

In the absence of damping and excitation, the system in Fig. 4-1 reduces to that shown in Fig. 4-2(*a*). Hence the equations of motion from Eq. (4-2) are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k+k_1 & -k \\ -k & k+k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4-9}$$

The equations are linear and homogeneous and are in the form of Eq. (D-47), App. D. Hence the solutions can be expressed as

$$x_1 = B_1 e^{st}$$
$$x_2 = B_2 e^{st} \tag{4-10}$$



(*a*) Vibratory system (*b*) Motions at principal modes

FIG. 4-2. *Modes of vibration*

where B_1, B_2, and s are constants. Since the system is undamped, it can be shown that the values of s are imaginary, $s = \pm j\omega$. By **Euler's** formula, $e^{\pm j\omega t} = \cos \omega t \pm j \sin \omega t$, and recalling that the x's are real, the solutions above must be harmonic and the general solution must consist of a number of harmonic components.

Assume one of the harmonic components is

$$\begin{aligned} x_1 &= A_1 \sin(\omega t + \psi) \\ x_2 &= A_2 \sin(\omega t + \psi) \end{aligned} \tag{4-11}$$

where A_1, A_2, and ψ are constants and ω is a natural frequency of the system. If the motions are harmonic, the choice of sine or cosine functions is arbitrary.

Substituting Eq. (4-11) in (4-9), dividing out the factor $\sin(\omega t + \psi)$, and rearranging, we have

$$\begin{aligned} (k + k_1 - \omega^2 m_1)A_1 - kA_2 &= 0 \\ -kA_1 + (k + k_2 - \omega^2 m_2)A_2 &= 0 \end{aligned} \tag{4-12}$$

which are homogeneous linear algebraic equations in A_1 and A_2. The determinant $\Delta(\omega)$ of the coefficients of A_1 and A_2 is **called** the characteristic determinant. If $\Delta(\omega)$ is equated to zero, we obtain the characteristic or the frequency equation of the system from which the values of ω are found, that is,

$$\Delta(\omega) = \begin{vmatrix} k + k_1 - \omega^2 m_1 & -k \\ -k & k + k_2 - \omega^2 m_2 \end{vmatrix} = 0 \tag{4-13}$$

From **linear** algebra, Eq. (4-12) possesses a solution only if the **determinant** $\Delta(\omega)$ is zero.

Expanding the determinant and rearranging, we get

$$\omega^4 - \left(\frac{k + k_1}{m_1} + \frac{k + k_2}{m_2}\right)\omega^2 + \frac{k_1 k_2 + k_1 k + k_2 k}{m_1 m_2} = 0 \tag{4-14}$$

which is quadratic in ω^2. This leads to two real and positive values* for ω^2. Calling them ω_1^2 and ω_2^2, the values of ω from Eq. (4-14) are $\pm\omega_1$ and $\pm\omega_2$. Since the solutions in Eq. (4-11) are harmonic, the negative signs for ω merely change the signs of the arbitrary constants and would not lead to new solutions. Hence the natural frequencies are ω_1 and ω_2.

The example shows that there are two natural frequencies in a **two**-degree-of-freedom system. Each of the solutions of Eq. (4-9) has two harmonic components at the frequencies ω_1 and ω_2, respectively. By

* Note that the values of s in Eq. (4-10) are $\pm j\omega_1$ and $\pm j\omega_2$ in order to have the periodic solutions assumed in Eq. (4-11). If ω^2 is not real and positive, it can be shown that the solutions by Eq. (4-10) would either diminish to zero or increase to infinity with increasing time.

superposition, the solutions from Eq. (4-11) are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \sin(\omega_1 t + \psi_1) + \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} \sin(\omega_2 t + \psi_2) \qquad \textbf{(4-15)}$$

where the A's and ψ's are arbitrary constants. The lower frequency term is called the fundamental and the others are the harmonics. Double subscripts are assigned to the amplitudes; the first subscript refers to the coordinate and the second to the frequency. For example, A,, is the amplitude of $x_1(t)$ at the frequency $\omega = \omega_2$.

The relative amplitudes of the harmonic components in Eq. (4-15) are defined in Eq. (4-12). Substituting ω_1 and ω_2 in Eq. (4-12) and rearranging, we obtain

$$\begin{aligned} \frac{A_{11}}{A_{21}} &= \frac{k}{k + k_1 - \omega_1^2 m_1} = \frac{k + k_2 - \omega_1^2 m_2}{k} = \frac{u_{11}}{u_{21}} \triangleq \frac{1}{u_1} \\ \frac{A_{12}}{A_{22}} &= \frac{k}{k + k_1 - \omega_2^2 m_1} = \frac{k + k_2 - \omega_2^2 m_2}{k} = \frac{u_{12}}{u_{22}} \triangleq \frac{1}{u_2} \end{aligned} \qquad (4\text{-}16)$$

where the u's are constants, defining the relative amplitudes of x_1 and x_2 at each of the natural frequencies ω_1 and ω_2. Thus, Eq. (4-15) becomes

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \end{bmatrix} A_{11} \sin(\omega_1 t + \psi_1) + \begin{bmatrix} 1 \\ u_2 \end{bmatrix} A_{12} \sin(\omega_2 t + \psi_2) \qquad \textbf{(4-17)}$$

where A_{11}, A,, ψ_1, and ψ_2 are the constants of integration, to be determined by the initial conditions. There are four constants because the system is described by two second-order differential equations. Note that (1) from the homogeneous equation in Eq. (4-12), only the ratios $1:u_1$ and $1:u_2$ can be found, and (2) the relative amplitudes at a given natural frequency are invariant, regardless of the initial conditions.

A principal or natural mode of vibration occurs when the entire system executes synchronous harmonic motion at one of the natural frequencies as illustrated in Fig. 4-2(b). For example, the first mode occurs if A,, $=0$ in Eq. (4-17), that is,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \end{bmatrix} A_{11} \sin(\omega_1 t + \psi_1) \quad \text{or} \quad \{x\} = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} p_1(t) \triangleq \{u\}_1 p_1(t) \qquad \textbf{(4-18)}$$

where $\{u\}_1$ is called a modal vector or eigenvector. Note that $\{u\}_1 \triangleq \{u_{11}\ u_{21}\} = \{1\ u_1\}$ as shown above. It represents the relative amplitude, or the mode shape, of the motions $x_1(t)$ and $x_2(t)$ at $w = w_1$. Hence a principal mode is specified by the modal vector at the given natural frequency. The quantity $p_1(t) = A_{11} \sin(\omega_1 t + \psi_1)$ is harmonic. It shows that the entire system executes synchronous harmonic motion at a principal mode. Similarly, the second mode occurs if A_{11} in Eq. (4-17) is zero, that is,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ u_2 \end{bmatrix} A_{12} \sin(\omega_2 t + \psi_2) \quad \text{or} \quad \{x\} = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} p_2(t) \triangleq \{u\}_2 p_2(t) \tag{4-19}$$

The modal vector for the second mode is $\{u\}_2$.

The harmonic functions of the motions $x_1(t)$ and $x_2(t)$ in Eq. (4-17) can be expressed as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ u_1 & u_2 \end{bmatrix} \begin{bmatrix} A_{11} \sin(\omega_1 t + \psi_1) \\ A_{12} \sin(\omega_2 t + \psi_2) \end{bmatrix} \triangleq \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \tag{4-20}$$

or

$$\{x\} = [u]\{p\} \tag{4-21}$$

where the modal matrix [u] is

$$[u] = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ u_1 & u_2 \end{bmatrix}$$

or

$$[u] = [u_{ij}] = [\{u\}_1 \quad \{u\}_2] \tag{4-22}$$

and $p_1(t) = A_{11} \sin(\omega_1 t + \psi_1)$ and $p_2(t) = A_{12} \sin(\omega_2 t + \psi_2)$. Note that in Eqs. (4-18) through (4-20) only the relative values in a modal vector can be defined, as shown in Eq. (4-16). A modal matrix [u] in Eq. (4-22) is simply a combination of the modal vectors. The actual motions {x) in Eq. (4-20) are specified by the constants A's and ψ's, which are determined by the initial conditions.

The vector {p) consists of a set of harmonic functions at the frequencies ω_1 and ω_2. The vector {p) is called the principal coordinates. Each principal coordinate $p_i(t)$ and its associated modal vector $\{u\}_i$ describe a mode of vibration as shown in Eqs. (4-18) and (4-19). Principal coordinates will be further discussed in Sec. 4-5 and Chap. 6. The coordinate transformation between the {x) and {p) coordinates is shown in Eq. (4-21).

The extension of the concept to n-degree-of-freedom systems is immediate. For example, a system may be described by the $\{x\} = \{x_1\, x_2 \ldots x_n\}$ coordinates. Analogous to Eq. (4-17), each of the motions $x_i(t)$ has n harmonic components. A principal mode occurs if the entire system executes synchronous harmonic motion at one of the natural frequencies. The corresponding principal coordinates is $\{p\} = \{p_1\, p_2 \ldots p_n\}$. The modal matrix [u] consists of n modal vectors $\{u\}_i$.

$$[u] = [\{u\}_1 \quad \{u\}_2 \cdots \{u\}_n] = [u_{ij}] \tag{4-23}$$

where $i, j = 1, 2, \ldots,$ n. The transformation between the $\{x\}$ and the $\{p\}$ coordinates is analogous to that in Eq. (4-21).

Example 1

Referring to Fig. 4-2(*a*), let $m_1 = m_2 = m$ and $k_1 = k_2 = k$. If the initial conditions are $\{x(0)\} = \{1 \quad 0\}$ and $\{\dot{x}(0)\} = \{0 \quad 0\}$, find the natural frequencies of the system and the displacement vector $\{x\}$.

Solution:

From Eq. (4-14), the natural frequencies are

$$\omega_1 = \sqrt{k/m} \qquad \text{and} \qquad \omega_2 = \sqrt{3k/m}$$

Substituting ω_1 and ω_2 in Eq. (4-16), we obtain $u_1 = 1$ and $u_2 = -1$. Hence the displacement vector $\{x\}$ from Eq. (4-20) is

$$\{x\} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A_{11} \sin(\omega_1 t + \psi_1) \\ A_{12} \sin(\omega_2 t + \psi_2) \end{bmatrix}$$

For the initial conditions $\{x(0)\} = \{1 \quad 0\}$, we get

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A_{11} \sin \psi_1 \\ A_{12} \sin \psi_2 \end{bmatrix}$$

Premultiplying the equation by the inverse $[u]^{-1}$ of $[u]$ gives

$$\begin{bmatrix} A_{11} \sin \psi_1 \\ A_{12} \sin \psi_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

or

$$A_{11} = \frac{1}{2 \sin \psi_1} \qquad \text{and} \qquad A_{12} = \frac{1}{2 \sin \psi_2}$$

For the initial conditions $\{\dot{x}(0)\} = \{0 \quad 0\}$, we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \omega_1 A_{11} \cos \psi_1 \\ \omega_2 A_{12} \cos \psi_2 \end{bmatrix}$$

Premultiplying the equation by the inverse $[u]^{-1}$ of $[u]$ gives

$$\begin{bmatrix} \omega_1 A_{11} \cos \psi_1 \\ \omega_2 A_{12} \cos \psi_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the A's and ω's are nonzero, we have $\cos \psi_1 = \cos \psi_2 = 0$. Let $\psi_1 = m\pi/2$ and $\psi_2 = n\pi/2$, where m and n are odd integers. It can be shown that the choice of m and n other than 1 will not lead to new solutions. Thus, $A_{11} = A_{12} = 1/2$.

From Eq. (4-17), we obtain

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos \sqrt{\frac{k}{m}}\, t + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos \sqrt{\frac{3k}{m}}\, t$$

The motions are plotted in Fig. 4-3 for $\sqrt{k/m} = 2m$. The example can be repeated for different initial conditions to show that the relative amplitudes of the principal modes remain unchanged. This is left as an exercise.

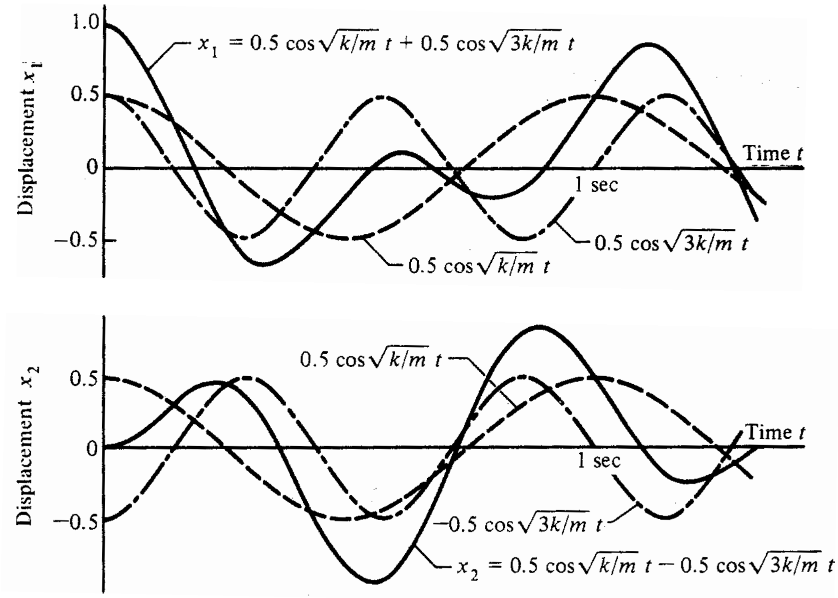


FIG. 4-3. Superposition of modes of vibration: Example 1.

Example 2. Natural modes

Find the initial conditions that would set a two-degree-of-freedom into its natural modes of vibration, that is, A_{11} or A_{12} in Eq. (4-17) becomes zero.

Solution:

From Eq. (4-20), we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ u_1 & u_2 \end{bmatrix} \begin{bmatrix} A_{11} \sin(\omega_1 t + \psi_1) \\ A_{12} \sin(\omega_2 t + \psi_2) \end{bmatrix}$$

Applying the initial conditions $\{x(0)\} = \{x_{10} \quad x_{20}\}$, we get

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ u_1 & u_2 \end{bmatrix} \begin{bmatrix} A_{11} \sin \psi_1 \\ A_{12} \sin \psi_2 \end{bmatrix}$$

Premultiplying the equation by the inverse $[u]^{-1}$ of $[u]$ gives

$$\begin{bmatrix} A_{11} \sin \psi_1 \\ A_{12} \sin \psi_2 \end{bmatrix} = \frac{1}{u_2 - u_1} \begin{bmatrix} u_2 & -1 \\ -u_1 & 1 \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

or

$$A_{11} = \frac{u_2 x_{10} - x_{20}}{(u_2 - u_1) \sin \psi_1} \qquad \text{and} \qquad A_{12} = \frac{x_{20} - u_1 x_{10}}{(u_2 - u_1) \sin \psi_2}$$

Similarly, using the initial conditions $\{\dot{x}(0)\} = \{\dot{x}_{10} \quad \dot{x}_{20}\}$, we get

$$A_{11} = \frac{u_2 \dot{x}_{10} - \dot{x}_{20}}{\omega_1 (u_2 - u_1) \cos \psi_1} \qquad \text{and} \qquad A_{12} = \frac{\dot{x}_{20} - u_1 \dot{x}_{10}}{\omega_2 (u_2 - u_1) \cos \psi_2}$$

The first mode occurs at ω_1 if $A_{12}=0$, that is,

$$x_{20}=u_1x_{10} \qquad \text{and} \qquad \dot{x}_{20}=u_1\dot{x}_{10}$$

In other words, the system will vibrate at its first mode if the initial conditions are $\{x_{10} \quad x_{20}\}=\{1 \quad u_1\}$ with zero initial velocities. Alternatively, the initial conditions can be $\{\dot{x}_{10} \quad \dot{x}_{20}\}=\{1 \quad u_1\}$ with zero initial displacements. Any combination of the above conditions would also set the system in its first mode. It is only necessary to set the initial values of $\{x\}$ and/or $\{\dot{x}\}$ to conform to their relative values for the first mode, as indicated by the corresponding modal vector $\{u\}_1=\{1 \quad u_1\}$.

Similarly, the second mode occurs when $A_{11}=0$, that is, $x_{20}=u_2x_{10}$ and $\dot{x}_{20}=u_2\dot{x}_{10}$. Any combination of these conditions will give the second mode.

Example 3. Vehicle suspension

An automobile is shown schematically in Fig. 4-4. Find the natural frequencies of the car body.

Solution:

An automobile has many degrees of freedom. Simplifying, we assume that the car moves in the plane of the paper and the motion consists of (1) the vertical motion of the car body, (2) the rotational pitching motion of the body about its mass center, and (3) the vertical motion of the wheels. Even then, the system bas more than two degrees of freedom.

When the excitation frequency due to the road roughness is high, the wheels move up and down with great rapidity but little of this motion is transmitted to the car body. In other words, the natural frequency of the car body is low and only the low frequency portion of the road roughness is being transmitted. (See Case 4 in Sec. 3-5.) Because of this large separation of natural frequencies between the wheels and the car body, the problem can be further simplified by neglecting the wheels as shown in Fig. 4-5.

Assuming small oscillations, the equations of motion in the $x(t)$ and $\theta(t)$ coordinates are

$$m i=\sum \text{(forces)},$$
$$m\ddot{x}=-k_1(x-L_1\theta)-k_2(x+L_2\theta)$$

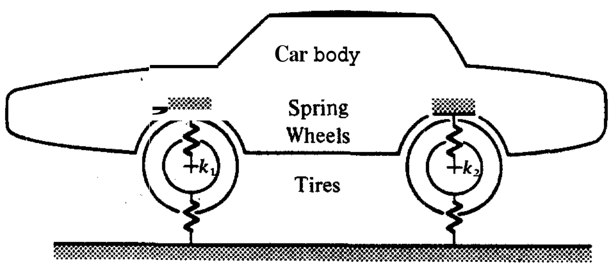


FIG. 4-4. Schematic *of an automobile.*

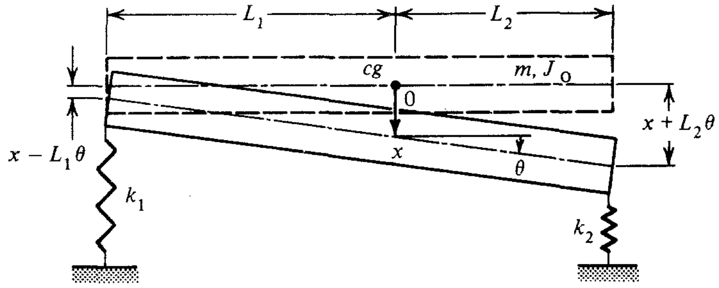


FIG. 4-5. *Simplified representation of an automobile body.*

and

$$J_0\ddot{\theta} = \sum (\text{moments}),$$

$$J_0\ddot{\theta} = k_1(x - L_1\theta)L_1 - k_2(x + L_2\theta)L_2$$

Rearranging, we obtain

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -(k_1L_1 - k_2L_2) \\ -(k_1L_1 - k_2L_2) & k_1L_1^2 + k_2L_2^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is of the same form as Eq. (4-9). The frequency equation from Eq. (4-13) is

$$\Delta(\omega) = \begin{vmatrix} k_1 + k_2 - \omega^2 m & k_2L_2 - k_1L_1 \\ k_2L_2 - k_1L_1 & k_1L_1^2 + k_2L_2^2 - \omega^2 J_0 \end{vmatrix} = 0$$

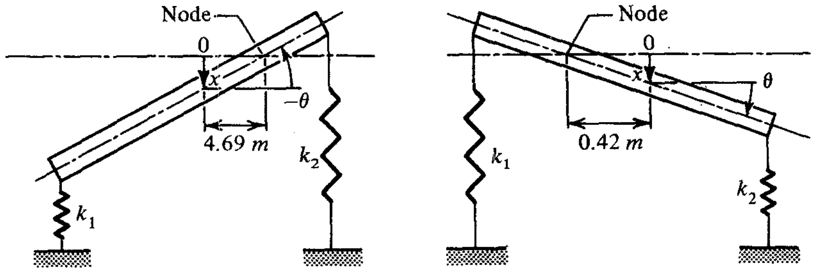
Expanding the determinant and solving the equation, we get

$$\omega_{1,2}^2 = \frac{1}{2}\left[\frac{k_1 + k_2}{m} + \frac{k_1L_1^2 + k_2L_2^2}{J_0} \mp \sqrt{\left(\frac{k_1 + k_2}{m} + \frac{k_1L_1^2 + k_2L_2^2}{J_0}\right)^2 - \frac{4k_1k_2(L_1 + L_2)^2}{mJ_0}}\right]$$

The natural frequencies are $\omega_1/2\pi$ and $\omega_2/2\pi$ Hz.

Example 4

A vehicle has a mass of 1,800 kg (4,000 lb) and a wheelbase of 3.6 m (140 in.). The mass center cg is 1.6 m (63 in.) from the front axle. The radius of gyration of the vehicle about *cg* is 1.4 m (55 in.). The spring constants of the front and the rear springs are 42 kN/m (240 lb_f/in.) and 48 kN/m (275 lb_f/in.), respectively. Determine (a) the natural frequencies, (b) the principal modes of vibration, and (c) the motion $x(t)$ and $\theta(t)$ of the vehicle.



(a) First mode: f = 1.09 Hz (b) Second mode: f = 1.50 Hz

FIG. 4-6. Principal modes of vibration of a car body (not to scale).

Solution:

(a) From the given data and the equations in Example 3, we have

$$\frac{k_1+k_2}{m}=50 \qquad \frac{k_1L_1-k_2L_2}{m}=-16.0$$

$$\frac{k_1L_1^2+k_2L_2^2}{J_0}=84.9 \qquad \frac{4k_1k_2(L_1+L_2)^2}{mJ_0}=16\,457$$

$$\omega_{1,2}^2=\tfrac{1}{2}[50+84.9\mp\sqrt{(50+84.9)^2-16\,457}]=\begin{cases}46.6\\88.3\end{cases}$$

$$\omega_{1,2}=\begin{cases}6.83\text{ rad/s}=1.09\text{ Hz}\\9.40\text{ rad/s}=1.50\text{ Hz}\end{cases}$$

(b) The amplitude ratios for the two modes of vibration are

$$\frac{X}{\Theta}=\frac{(k_1L_1-k_2L_2)/m}{(k_1+k_2)/m-\omega_{1,2}^2}=\frac{-16}{50-\omega_{1,2}^2}=\begin{cases}-4.69\text{ m/rad}\\\ \ 0.42\text{ m/rad}\end{cases}$$

The two principal modes of vibration are shown schematically in Fig. 4-6. The mode shape at 1.09 Hz is $\{X \quad \Theta\}=\{1 \quad -1/4.69\}$. Thus, when $x(t)$ is positive, $\theta(t)$ is negative from the assumed direction of rotation. When $x(t)=1$ m, $\theta(t)=-1/4.69$ rad, that is, the *node* is 4.69 m from the cg of the car body. Similarly, at 1.50 Hz, the mode shape is $\{X \quad \Theta\}=\{1 \quad 1/0.42\}$.

(c) From Eq. (4-17), the $x(t)$ and $\theta(t)$ motions are

$$\begin{bmatrix}x\\\theta\end{bmatrix}=\begin{bmatrix}1\\-1/4.69\end{bmatrix}A_{11}\sin(6.83t+\psi_1)+\begin{bmatrix}1\\1/0.42\end{bmatrix}A_{12}\sin(9.40t+\psi_2)$$

where A_{11}, A_{12}, ψ_1, and ψ_2 are the constants of integration.

Example 5. A three-degree-of-freedom system

A torsional system with three degrees of freedom is shown in Fig. 4-7. **(a)** Determine the equations of motion and the frequency equation. (b) If

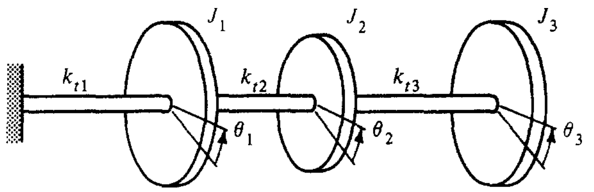


FIG. 4-7. *A three-degree-of-freedom torsional system; Example 5.*

$J_1 = J_2 = J_3 = J$ and $k_{t1} = k_{,,} = k_{t3} = k_{,,}$ find the natural frequencies and the equation for the displacement $\{\theta\}$.

Solution:

(a) From Newton's second law, the equations of motion are

$$\begin{aligned} J_1\ddot{\theta}_1 &= -k_{t1}\theta_1 - k_{t2}(\theta_1 - \theta_2) \\ J_2\ddot{\theta}_2 &= -k_{t2}(\theta_2 - \theta_1) - k_{t3}(\theta_2 - \theta_3) \\ J_3\ddot{\theta}_3 &= -k_{t3}(\theta_3 - \theta_2) \end{aligned} \qquad \textbf{(4-24)}$$

Substituting $\theta_i = \Theta_i \sin(\omega t + +)$, for $i = 1, 2$, and 3, in **Eq.** (4-24), factoring out the $\sin(\omega t + \psi)$ term, and rearranging, we have

$$\begin{aligned} (k_{t1} + k_{t2} - \omega^2 J_1)\Theta_1 - k_{t2}\Theta_2 &= 0 \\ -k_{t2}\Theta_1 + (k_{t2} + k_{t3} - \omega^2 J_2)\Theta_2 - k_{t3}\Theta_3 &= 0 \\ -k_{t3}\Theta_2 + (k_{t3} - \omega^2 J_3)\Theta_3 &= 0 \end{aligned} \qquad \textbf{(4-25)}$$

The frequency equation is obtained by equating the determinant $\Delta(\omega)$ of the coefficients of Θ_1, Θ_2, and Θ_3 to zero.

$$\Delta(\omega) = \begin{vmatrix} k_{t1} + k_{t2} - \omega^2 J_1 & -k_{t2} & 0 \\ -k_{t2} & k_{t2} + k_{t3} - \omega^2 J_2 & -k_{t3} \\ 0 & -k_{t3} & k_{t3} - \omega^2 J_3 \end{vmatrix} = 0$$

(b) If $J_1 = J_2 = J_3 = J$ and $k_{t1} = k_{t2} = k_{t3} = k_t$, the frequency equation is

$$\omega^6 - 5\left(\frac{k_t}{J}\right)\omega^4 + 6\left(\frac{k_t}{J}\right)^2\omega^2 - \left(\frac{k_t}{J}\right)^3 = 0$$

The roots of the equation are $0^2 = 0.198(kJJ)$, $1.55(k_t/J)$, and $3.25(k_t/J)$. The corresponding frequency vector $\{f\}$ is

$$\{f\} = \frac{1}{2\pi}\cdot\sqrt{\frac{k_t}{J}}\begin{bmatrix} \sqrt{0.198} \\ \sqrt{1.55} \\ \sqrt{3.25} \end{bmatrix} = \sqrt{\frac{k_t}{J}}\begin{bmatrix} 0.071 \\ 0.198 \\ 0.288 \end{bmatrix} \text{Hz}$$

From Eq. (4-25), the amplitude ratios are

$$\frac{\Theta_1}{\Theta_2} = \frac{k_{t2}}{k_{t1} + k_{t2} - \omega^2 J_1} \quad \text{and} \quad \frac{\Theta_2}{\Theta_3} = \frac{k_{t3} - \omega^2 J_3}{k_{t3}}$$

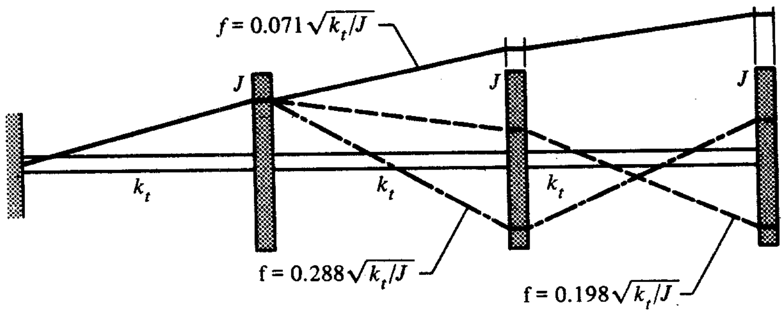


FIG. 4-8. *Principal modes of vibration; amplitudes plotted normal to axis of rotation; Example 5.*

Thus, a modal vector $\{\Theta_1 \quad \Theta_2 \quad \Theta_3\}_i$ can be calculated for each of the natural frequencies ω_i. The modal matrix $[\theta_{ij}]$, for $i, j = 1, 2,$ and 3, is formed from a combination of the modal vectors $\{\theta\}_i$ as in Eq. (4-21).

$$[\theta_{ij}] = [\{\theta\}_1 \quad \{\theta\}_2 \quad \{\theta\}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1.802 & 0.445 & -1.247 \\ 2.25 & -0.802 & 0.555 \end{bmatrix}$$

where $\{\theta\}_1$, $\{\theta\}_2$, and $\{\theta\}_3$ are the modal vectors for the frequencies $\omega_1 = \sqrt{0.198k_t/J}$, $\omega_2 = \sqrt{1.55k_t/J}$, and $\omega_3 = \sqrt{3.25k_t/J}$, respectively. The principal modes of vibration are illustrated in Fig. 4-8.

By superposition of the principal modes, the motions $\{\theta\}$ of the rotating disks are

$$\{\theta\} = \begin{bmatrix} 1 \\ 1.802 \\ 2.25 \end{bmatrix} \Theta_{11} \sin(\omega_1 t + \psi_1) + \begin{bmatrix} 1 \\ 0.445 \\ -0.802 \end{bmatrix} \Theta_{12} \sin(\omega_2 t + \psi_2) + \begin{bmatrix} 1 \\ -1.247 \\ 0.555 \end{bmatrix} \Theta_{13} \sin(\omega_3 t + \psi_3)$$

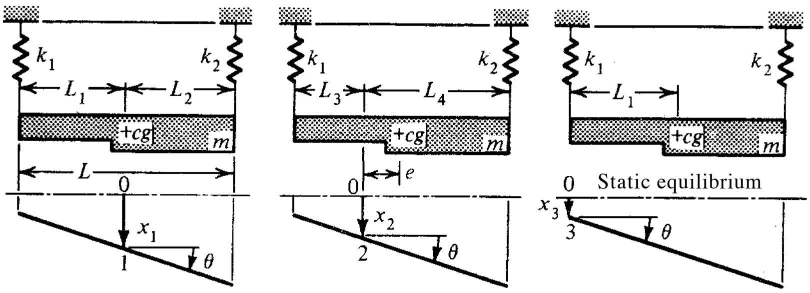
The Θ's and ψ's are to be determined by the initial conditions.

4-4 GENERALIZED COORDINATES AND COORDINATE COUPLING

The general form of the equations of motion of a two-degree-of-freedom system is shown in Eq. (4-4). For undamped free vibration, we have

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4-26}$$

The system is described by the coordinates x_1 and x_2, which are the elements of the displacement vector $\{x\}$. The coupling terms in the equations are m_{12}, m_{21}, k_{12}, and k_{21}. We shall show that the values of the



(a) Static coupling (b) Dynamic coupling (c) Static and dynamic coupling

FIG. 4-9. ***Generalized coordinates and coordinate coupling: a two-degree-of-freedom system described by the*** (x,,θ), (x_2,0), ***and*** (x_3,θ) ***coordinates.***

elements in the matrices M and K are dependent on the coordinates selected for the system description.

A vibratory system can be described by more than one set of independent spatial coordinates, each of which can be called a set of generalized coordinates. We often use the displacements from the static equilibrium positions of the masses and the rotations about the mass centers for the coordinates. This choice is convenient, but, nonetheless, arbitrary. We shall describe the system in Fig. 4-9 by the displacement vectors $\{x_1 \quad \theta\}$, $\{x_2 \quad \theta\}$, and $\{x_3 \quad \theta\}$.

Referring to Fig. 4-9(a) and assuming small oscillations, the equations of motion in the (x_1,θ) coordinates are

$$m\ddot{x}_1 = -k_1(x_1 - L_1\theta) - k_2(x_1 + L_2\theta)$$
$$J_1\ddot{\theta} = k_1(x_1 - L_1\theta)L_1 - k_2(x_1 + L_2\theta)L_2$$

Rearranging, we obtain

$$\begin{bmatrix} m & 0 \\ 0 & J_1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -(k_1L_1-k_2L_2) \\ -(k_1L_1-k_2L_2) & (k_1L_1^2+k_2L_2^2) \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \textbf{(4-27)}$$

The coupling term, $-(k_1L_1 - k_2L_2)$, occurs only in the stiffness matrix, and the system is said to be statically or elastically coupled (see Examples 3 and 4). If a static force is applied through the cg at point 1, the body will rotate as well as translate in the x_1 direction. Conversely, if a torque is applied at point 1, the body will translate as well as rotate in the 8 direction.

The same system is described by the (x_2,θ) coordinates in Fig. 4-9(b). The distance e is selected to give $k_1L_3 = k_2L_4$. If a static force is applied at point 2 to cause a displacement x_2, the body will not rotate. Hence no static coupling is anticipated in the equations of motion. During vibration, however, the inertia force $m\ddot{x}_2$ through cg will create a moment $m\ddot{x}_2e$

about point 2, tending to rotate the body in the θ direction. Conversely, a rotation θ about point 2 will give a displacement $e\theta$ at cg and therefore a force $me\ddot{\theta}$ in the x_2 direction. Hence, dynamic coupling is anticipated in the equations.

The equations of motion in the (x_2,θ) coordinates are

$$m\ddot{x}_2 = -k_1(x_2 - L_3\theta) - k_2(x_2 + L_4\theta) - me\ddot{\theta}$$
$$J_2\ddot{\theta} = k_1(x_2 - L_3\theta)L_3 - k_2(x_2 + L_4\theta)L_4 - me\ddot{x}_2$$

$$\begin{bmatrix} m & me \\ me & J_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1+k_2 & 0 \\ 0 & k_1L_3^2 + k_2L_4^2 \end{bmatrix}\begin{bmatrix} x_2 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \textbf{(4-28)}$$

The coupling terms are associated with the inertia forces and the system is said to be dynamically, or inertia, coupled.

Lastly, let the same system be described by the (x_3,θ) coordinates as shown in Fig. 4-9(c). It can be shown that the equations of motion are

$$\begin{bmatrix} m & mL_1 \\ mL_1 & J_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_3 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1+k_2 & k_2L \\ k_2L & k_2L^2 \end{bmatrix}\begin{bmatrix} x_3 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \textbf{(4-29)}$$

For the (x_3,θ) description, the equations are statically and dynamically coupled.

Note that (1) the choice of coordinates for the system description is a mere convenience, (2) the system will vibrate in its own natural way regardless of the coordinate description, (3) the equations for one coordinate description can be obtained from those for other descriptions, and (4) coupling in the equations is not an inherent property of the system, such as natural frequencies.

The examples above show that the matrices M and K are symmetric, that is, $m_{12} = m_{21}$ and $k_{12} = k_{21}$. Symmetry is assured if the deflections are measured from a fixed position in space. This can be deduced from Maxwell's reciprocity theorem in Sec. 4-9. Let us select a set of generalized coordinates based on relative deflections to illustrate the nonsymmetic matrices in the equations of motion.

Example 6

Consider the system shown in Fig. 4-2 and assume the generalized coordinates $q_1 = x_1$ and $q_2 = x_2 - x_1$, that is, q_2 is proportional to the spring force due to k. Find the equations of motion of the system.

Solution:

The coordinates $\{q\}$ and $\{x\}$ are related by

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Replacing the displacement and acceleration vectors $\{x\}$ and $\{\ddot{x}\}$ by $\{q\}$ and $\{\ddot{q}\}$ in Eq. (4-9) for the same system, we get

$$\begin{bmatrix} m_1 & 0 \\ m_2 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k \\ k_2 & k+k_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

4-5 PRINCIPAL COORDINATES

It was shown in the last section that the elements of the matrices M and K depend on the coordinates selected for the system description. It is possible to select a particular set of coordinates, called the principal coordinates, such that there is no coupling terms in the equations of motion, that is, the matrices M and K become diagonal matrices. Hence each of the uncoupled equations can be solved independently. In other words, when the system is described in terms of the principal coordinates, the equations of motion are uncoupled, and the modes of vibration are mathematically separated. Thus, each of the uncoupled equations can be solved independently, as if for systems with one degree of freedom.

Assume an undamped two-degree-of-freedom system is uncoupled by the principal coordinates $\{p\}$. The corresponding equations of motion from Eq. **(4-4)** are

$$\begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \textbf{(4-30)}$$

Expanding the equations gives

$$m_{11}\ddot{p}_1 + k_{11}p_1 = 0$$
$$m_{22}\ddot{p}_2 + k_{22}p_2 = 0$$

The solutions of the equations are

$$p_1 = A_{11}\sin(\omega_1 t + \psi_1)$$
$$p_2 = A_{12}\sin(\omega_2 t + \psi_2) \qquad \textbf{(4-31)}$$

where $\omega_1^2 = k_{11}/m_{11}$, $\omega_2^2 = k_{22}/m_{22}$, and the A's and ψ's are constants. Evidently, each of the solutions above represents a mode of vibration as discussed in **Sec.** 4-3. At a given mode, the system resembles an independent one-degree-of-freedom system.

Now, assume the same system is described by the generalized coordinates $\{q\}$ and the equations of motion are coupled. From Eq. **(4-17),** the motions in the {q) coordinates are

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} A_{11} \sin(\omega_1 t + \psi_1) + \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} A_{21} \sin(\omega_2 t + \psi_2) \qquad \textbf{(4-32)}$$

where $\{u_{11} \quad u_{21}\}$ and $\{u_{12} \quad u_{22}\}$ are the modal vectors for the frequencies ω_1 and ω_2, respectively.

Substituting Eq. **(4-31)** in **(4-32)** and simplifying, we get

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \qquad \textbf{(4-33)}$$

or

$$\{q\} = [u]\{p\} \qquad \text{and} \qquad \{p\} = [u]^{-1}\{q\} \qquad \textbf{(4-34)}$$

where $[u]^{-1}$ is the inverse of the modal matrix [u], as defined in Eq. **(4-22).** The transfornnation between the $\{p\}$ and the {q) coordinates in Eq. **(4-34)** is identical to that shown in Eq. **(4-21).***

The discussion implies that the equations of motion can be uncoupled by means of a coordinate transformation. In other words, given the coupled equations in the {q) coordinates, the equations can be uncoupled by substituting {p) for {q) as shown in Eq. **(4-34).** This can be done, but the general theory **in Sec. 6-4** will be needed. In the mean time, we shall illustrate with another example and then show the general **technique** in the next section.

Example *7*

Determine the principal coordinates for the **system** shown in Fig. 4-2(a) if $m_1 = m_2 = m$ and $k_1 = k_2 = k$.

Solution:

From Example 1, we have $u_1 = 1$ and $u_2 = -1$. Hence *Eq.* **(4-33)** becomes

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The transformation above indicates that, if $p_1 = \frac{1}{2}(x_1 + x_2)$ and $p_2 = \frac{1}{2}(x_1 - x_2)$, the equations of motion in the $\{p\}$ coordinates are uncoupled. Let us further examine this statement.

The equations of motion for the same system from Eq. (4-9) are

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 + 2kx, - kx, = 0$$

* We assume that the matrices M and K in the generalized coordinates are symmetric. An inherent symmetry in the system can he assumed. We shall not discuss nonsymmetric matrices as illustrated in Example 6.

Adding and subtracting the equations, we obtain

$$m(\ddot{x}_1+x,)+k(x_1+x_2)=0 \qquad \text{or} \qquad m\ddot{p}_1+kp_1=0$$
$$m(\ddot{x}_1-\ddot{x}_2)+3k(x_1-x_2)=0 \qquad\qquad m\ddot{p}_2+3kp_2=0$$

Again, the equations are uncoupled if we define $p_1=(x,+x,)$ and $p_2=(x_1-x_2)$. Since the amplitudes of oscillation are arbitrary, the factor $(\frac{1}{2})$ between the two definitions of p_1 and p_2 in the problem is secondary.

4-6 MODAL ANALYSIS: TRANSIENT VIBRATION OF *UNDAMPED* SYSTEMS

Consider the steps to solve the equations of motion of an undamped system. From Eq. **(4-5),** we get

$$M\{\ddot{q}\}+K\{q\}=\{Q(t)\} \tag{4-35}$$

(1) The equations can be uncoupled by means of the modal matrix [u] and expressed in the principal coordinates $\{p\}$ as shown in Example 7. (2) Each of the uncoupled equations can be solved as an independent **one-**degree-of-freedom system. (3) Applying the coordinate transformation in Eq. **(4-34),** the solution can be expressed in the $\{p\}$ or $\{q\}$ coordinates as desired. The steps enumerated are conceptually simple. Except for the formula to uncouple the equations of motion, we have the necessary information for the modal analysis of transient vibration of undamped systems. For systems with more than two **degrees** of freedom, however, computer solutions, as illustrated in Chap. 9, are manditory to alleviate the numerical computations.

The. modal matrix of undamped systems can be found by the method described in the previous sections. From Eq. **(4-35),** the equations of motion of an unforced system are

$$M\{\ddot{q}\}+K\{q\}=\{0\} \tag{4-36}$$

A principal mode occurs if the entire system executes synchronous .harmonic motion at a natural frequency ω. Thus, the acceleration of q_i is $\ddot{q}_i=-0^2q_i$, or $\{\ddot{q}\}=\{-0^2q\}=-\omega^2\{q\}$. Substituting $-\omega^2\{q\}$ for $\{\ddot{q}\}$ in Eq. (4-36) and simplifying, we get

$$[-o^2M+K]\{q\}=\{0\} \tag{4-37}$$

Since the system is at a principal mode, the displacement vector $\{q\}$ is also a modal vector at the natural **frequency** o. In other words, $\{q\}$ in Eq. *(4-37)* gives the relative amplitude of vibration of the masses of the system at the given natural frequency.

Since Eq. (4-37) is a set of homogeneous algebraic equations, it possesses a solution only if the characteristic determinant $\Delta(\omega)$ is zero;

that is,

$$\Delta(\omega) = |K - \omega^2 M| = 0 \tag{4-38}$$

This is the characteristic of the frequency equation, which may be compared with Eq. (4-13). Previously, the frequency equation was solved by hand calculations as shown in Eq. (4-14). Computer solutions, however, are necessary for systems with more than two degrees of freedom. Likewise, instead of solving for the modal vector by hand calculations as shown in Eq. (4-16), computers can be used to solve for {q} in Eq. (4-37). A modal vector is found for each natural frequency. The modal matrix [u] is formed from a combination of the modal vectors as shown in Eq. (4-23).

Example *8*

Find the coefficients of the frequency equation for the system shown in Fig. 4-10(*a*).

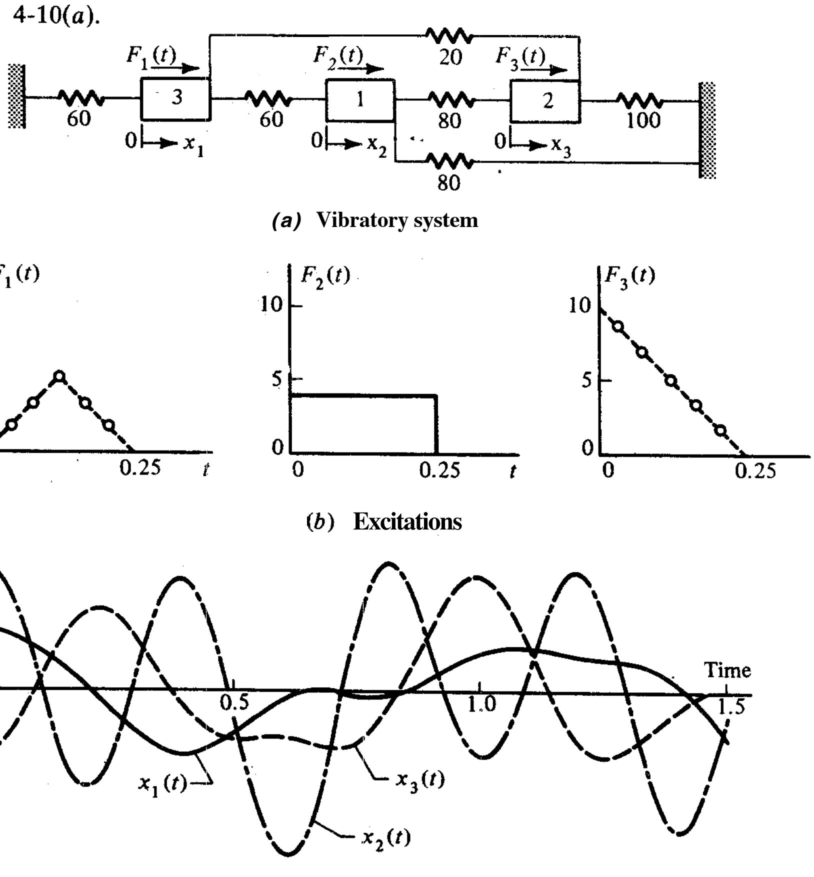


FIG. 4-10. *Transient vibration of undamped system: Examples 8 to 10.*

Solution:

Applying Newton's second law to each of the masses of the system, we have

$$3\ddot{x}_1 = -60x_1 - 60(x_1 - x_2) - 20(x_1 - x_3) + F_1(t)$$
$$1\ddot{x}_2 = -80x_2 - 60(x_2 - x_1) - 80(x_2 - x_3) + F_2(t)$$
$$2\ddot{x}_3 = -100x_3 - 20(x_3 - x_1) - 80(x_3 - x_2) + F_3(t)$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 140 & -60 & -20 \\ -60 & 220 & -80 \\ -20 & -80 & 200 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} \quad \text{(4-39)}$$

or

$$M\{\ddot{x}\} + K\{x\} = \{F(t)\}$$

The program COEFF to find the coefficients of the frequency equation is listed in Fig. **4-11(*a*)**. The values of the matrices M and K are entered (READ) and verified (WRITE) in Par. #I. The computations in Par. **#II** first change Eq. (4-36) into the form

$$\{\ddot{q}\} + M^{-1}K\{q\} = \{0\}$$

The subroutine SINVS is called to find the inverse M^{-1} **(MINVS)** of M. The subroutine SMPLY performs the matrix multiplication **MINVS*K** = H, where $H = M^{-1}K$ is the dynamic matrix. The subroutine **$COEFF** is called to give the coefficients of the frequency equation.

The print-out is listed in Fig. **4-11(*b*)**. We first give the command

MERGE COEFF, **$COEFF, $INVS, $MPLY, $SUBN**

to merge the main program COEFF with the necessary subroutines. The subroutine **$SUBN** is for the matrix substitutions in the calculations. When the computer is READY, the command RUN is given to start the program. The data entries are self-explanatory. The **frequency equation** is

$$1 - 45.685 \times 10^{-3}\omega^2 + 515.94 \times 10^{-6}\omega^4 - 1.4071 \times 10^{-6}\omega^6 = 0$$

Example 9

Assume the roots of the frequency equation in Example 8 are $\omega^2 = 33.23$, 86.67, and 246.8. Find the **modal** matrix of the system.

Solution:

For the given values of M and K and $\omega^2 = 33.23$, the direct application of Eq. (4-37) gives

$$\begin{bmatrix} 140 - 33.23 \times 3 & -60 & -20 \\ -60 & 220 - 33.23 \times 1 & -80 \\ -20 & -80 & 200 - 33.23 \times 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \{0\}$$

It can **be** shown that the solution of the homogeneous algebraic equation is $\{x\} = \{2.172 \quad 1.126 \quad 1.0\} = \{u\}_1$, where $\{u\}_1$ is the modal vector for $\omega^2 =$

```
COEFF

■  *** COEFFICIENTS OF CHARACTERISTIC POLYNOMIAL ***
*      SUBROUTINES REQUIRED: (1) SCOEFF (2) SINVS (3) $MPLY (4) $SUBN
■
*  *** #I. FORMAT AND INPUT. ***
   REAL*8  C(12), H(10,10), K(10,10), M(10,10), MINVS(10,10)
80 FORMAT (' COEFFICIENTS OF CHARACTERISTIC POLYNOMIAL.',/)
81 FORMAT (' ENTER: (1) ORDER OF MATRICES M & K   ** N <= 10',/,8X,%
   '(2) MATRIX-M BY ROW',/,8X,'(3) MATRIX-K BY ROW')
82 FORMAT (' N = ', I3)
83 FORMAT (5D14.5)
84 FORMAT (' *** IS THIS CORRECT?   1 = YES;  2 = NO.')
IS FORMAT (' CHARACTERISTIC POLYNOMIAL:',/, %
   '   F(X) = CO + C1*X + C2*X+2 + C3*X+3 + . . + CN*X+N',/, %
   '   THE VALUES OF CO TO CN ARE:-',/)
   WRITE (6,80)
10 WRITE (6,81)
   READ (5,*) N, ((M(I,J), J=1,N), I=1,N), ((K(I,J), J=1,N), I=1,N)
   WRITE (6,82) N
   DO 40 I=1.N
40   WRITE (6,83) (M(I,J), J=1,N)
   DO 41 I-1.N
41   WRITE (6,83) (K(I,J), J=1,N)
   WRITE (6,84)
   READ (5,*) IANS
   IF (IANS = 2) GOTO 10
   NP2 = N + 2
*
   *** #II. CALCULATIONS AND OUTPUT. ***
   CALL SINVS (M, MINVS, N)
   CALL $MPLY (MINVS, K, H, N)
   CALL $COEFF (H, C, N)
   WRITE (6.85:
   WRITE (6.83) (C(I), I=2.NP2)
   STOP
   END
```

(a) Main program

```
MERGE COEFF, $COEFF, SINVS, $MPLY, $SUBN
RUDY

RUN

COEFFICIENTS OF CHARACTERISTIC POLYNOMIAL.

ENTER: (1) ORDER OF MATRICES M & K   ** N <= 10
       (2) MATRIX-M BY RCW
       (3) MATRIX-K BY BOW
?3   3 0 0  0 1 0  0 0 2  140 -60 -20  -60 220 -80  -20 -80 100

N =   3
  0.30000D+01   0.0           0.0
  0.0           0.10000D+01   0.0
  0.0           0.0           0.20000D+01
  0.14000D+03  -0.60000D+02  -0.20000D+02
 -0.60000D+02   0.22000D+03  -0.80000D+02
 -0.20000D+02  -0.80000D+02   0.20000D+03
*** IS THIS CORRECT?   1 = YES;  2 = YO.
71

CHARACTERISTIC POLYNOMIAL:
  F(X) = CO + C1*X + C2*X+2 + C3*X+3 + . . + CN*X+N
  THE VALUES OF CO TO CN ARE:-

  0.10000D+01  -0.45685D-01   0.51594D-03  -0.14071D-05
```

(b) Print-out

FIG. 4-11. Program to **find** *coefficients of frequency equation: Example 8.*

33.23. Similarly, for $\omega^2 = 86.67$ and 246.8, the modal vectors are $\{u\}_2 = (-0.381 \;\; 0.429 \;\; 1.0)$ and $\{u\}_3 = \{0.342 \;\; -3.755 \;\; 1.0\}$, respectively. Hence the modal matrix is

$$[u] = [\{u\}_1 \;\; \{u\}_2 \;\; \{u\}_3] = \begin{bmatrix} 2.172 & -0.381 & 0.342 \\ 1.126 & 0.429 & -3.755 \\ 1.0 & 1.0 & 1.0 \end{bmatrix}$$

The equations of motion of undamped systems can be uncoupled by the orthogonal relations, which will be derived in **Sec. 6-4.** These are

$$\begin{aligned} [u]^T M[u] &= \lceil M \rfloor \\ [u]^T K[u] &= \lceil K \rfloor \end{aligned} \tag{4-40}$$

where $[u]^T$ is the transpose of the modal matrix $[u]$ and $\lceil M \rfloor$ and $\lceil K \rfloor$ are diagonal matrices. Substituting $\{q\} = [u]\{p\}$ from **Eq. (4-34) in (4-35),** we have

$$M[u]\{\ddot{p}\} + K[u]\{p\} = \{Q(t)\}$$

Premultiplying this by the transpose $[u]^T$ of $[u]$ gives

$$[u]^T M[u]\{\ddot{p}\} + [u]^T K[u]\{p\} = [u]^T\{Q(t)\}$$

or

$$\lceil M \rfloor\{\ddot{p}\} + \lceil K \rfloor\{p\} \triangleq \{N(t)\} \tag{4-41}$$

where $\{N(t)\} = [u]^T\{Q(t)\}$ is the excitation associated with the $\{p\}$ coordinates. Since $\lceil M \rfloor$ and $\lceil K \rfloor$ are **diagonal, Eq. (4-41)** gives the uncoupled equations.

Example 10

Describe a procedure to find the transient response of the system in Fig. **4-10(*a*).** Assume the excitation for each of the masses is as shown in Fig. **4-10(*b*).** The initial conditions are $\{x(0)\} = \{1 \quad 2 \quad -1\}$ and $\{\dot{x}(0)\} = \{-3 \quad 4 \quad 1\}$.

Solution:

We shall first show the orthogonal relations in Eq. (4-40) and then describe the procedure. Substituting the values of M, K, and $[u]$ from Example 9 in **Eq. (4-40)** yields

$$[u]^T M[u] = \begin{bmatrix} 2.172 & 1.126 & 1.0 \\ -0.381 & 0.429 & 1.0 \\ 0.342 & -3.755 & 1.0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2.172 & -0.381 & 0.342 \\ 1.126 & 0.429 & -3.755 \\ 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$$= \begin{bmatrix} 174.3 & 0 & 0 \\ 0 & 2.619 & 0 \\ 0 & 0 & 16.45 \end{bmatrix} = \lceil M \rfloor.$$

Similarly, it can be shown that the K matrix gives

$$[u]^T K[u] = \begin{bmatrix} 579.0 & 0 & 0 \\ 0 & 227.0 & 0 \\ 0 & 0 & 4059 \end{bmatrix} = \lceil K \rfloor$$

Hence, from Eq. (4-41), the uncoupled equations of motion are

$$1.743\ddot{p}_1 + 579p_1 = N_1(t)$$
$$2.619\ddot{p}_2 + 227p_2 = N_2(t) \qquad (4\text{-}42)$$
$$16.45\ddot{p}_3 + 4059p_3 = N_3(t)$$

From Eq. (4-34), the initial conditions expressed in terms of the principal coordinates $\{p\}$ are

$$\{p(0)\} = [u]^{-1}\{x(0)\} \quad \text{and} \quad \{\dot{p}(0)\} = [u]^{-1}\{\dot{x}(0)\} \qquad (4\text{-}43)$$

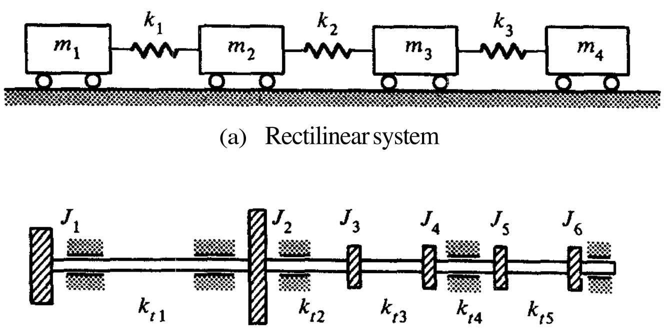
The uncoupled equations in Eq. *(4-42)* and the initial conditions from Eq. *(4-43)* are used to solve the problem in terms of the principal coordinates $\{p\}$. The resppnse of the masses in the $\{x\}$ coordinates are obtained by means of the transformation $\{x\}=[u]\{p\}$ in *Eq.* (4-34).

Since the excitations $\{F(t)\}$ shown in Fig. 4-10(*b*) are arbitrary, they are quantized and assumed to have constant values for each time interval $At = 0.05$. The solutions are plotted in Fig. 4-10(*c*). Although the procedure is conceptually simple, computer solutions are mandatory. The problem is solved by the program **TRESPUND** shown in Sec. *9-8*.

4-7 SEMIDEFINITE SYSTEMS

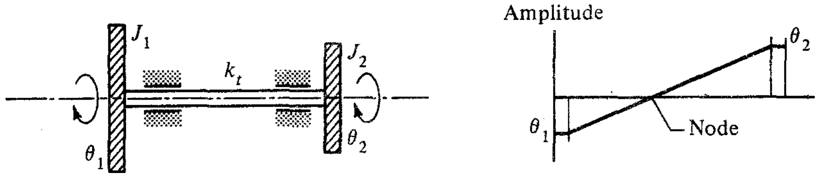
A special case of practical importance occurs when a root of the frequency equation vanishes. When a natural frequency is zero, there is no relative motion in the system. The system can move as a rigid body and is called *semidefinite.*

Two semidefinite systems are shown in Fig. 4-12. The rectilinear system consists of a number of masses coupled by springs. It may be used to represent the vibration of a train. The rotational system may represent a rotating machine, such as a diesel engine for marine propulsion. One of the disks may represent the propeller, another disk the flywheel, and the remaining disks the rotating and the equivalent reciprocating parts of the engine.



(b) Rotational system

FIG. *4-12. Semidefinite systems.*



(a) Vibratory system (b) Principal mode

FIG. 4-13. *Schematic of a motor-generator set.*

Let an electrical motor-generator set be represented by a two-disk system shown in Fig. 4-13(*a*). The rotors J_1 and J_2 are connected by a shaft of spring constant k_t. Summing the torques for each rotor about the shaft axis, the equations of motion are

$$J_1\ddot{\theta}_1 = -k_t(\theta_1 - \theta_2)$$

$$J_2\ddot{\theta}_2 = -k_t(\theta_2 - \theta_1)$$

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_t & -k_t \\ -k_t & k_t \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \textbf{(4-44)}$$

This is of the same form as Eq. (4-9). Hence the frequency equation from Eq. (4-13) is

$$\Delta(\omega) = \begin{vmatrix} k_t - \omega^2 J_1 & -k_t \\ -k_t & k_t - \omega^2 J_2 \end{vmatrix} = 0 \quad \textbf{(4-45)}$$

Expanding the determinant and dividing by $J_1 J_2$, we get

$$\omega^2\left[\omega^2 - \left(\frac{k_t}{J_1} + \frac{k_t}{J_2}\right)\right] = 0 \quad \textbf{(4-46)}$$

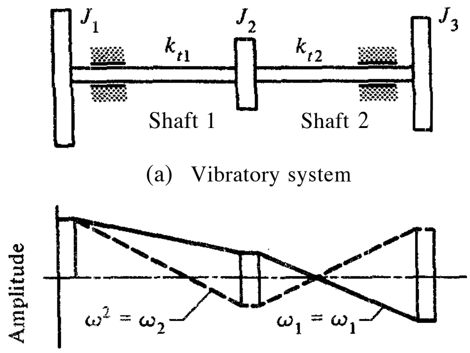
The roots of the equation are $\omega^2 = 0$ and $\omega^2 = k_t(1/J_1 + 1/J_2)$.

Following the procedure in Eq. (4-16), the relative amplitude of the disks at the principal modes are

$$\frac{\Theta_1}{\Theta_2} = \frac{k_t}{k_t - \omega^2 J_1} = \frac{k_t - \omega^2 J_2}{k_t} = \begin{cases} 1 & \text{for } \omega = 0 \\ -J_2/J_1 & \text{for } \omega = \omega_1 \end{cases} \quad \textbf{(4-47)}$$

where $\omega_1 = \sqrt{k_t(1/J_1 + 1/J_2)}$ as indicated in Eq. (4-46). When $\omega = 0$ and $\Theta_1/\Theta_2 = 1$, the two disks have identical angular displacements. Since there is no relative displacement between the disks, the shaft is not stressed and the assembly rotates as a rigid body. This is called the zero mode. When $\omega = \omega_1$, the two disks oscillate in opposite directions and the mode shape is $\{\Theta_1 \quad \Theta_2\} = \{J_2 \quad -J_1\}$ as illustrated in Fig. 4-13(*b*).

To extend the theory to systems with more than two degrees of freedom, consider the three-disk assembly in Fig. 4-14(*a*). The system could be used to represent, to the first approximation, a multidegree-of-freedom system, such as a diesel engine for marine propulsion. From



(a) Vibratory system

(*b*) Principal modes: $\omega_1 < \omega_2$

FIG. *4-14. A semidefinite system.*

Newton's second law, the equations of motion of the disks are

$$
\begin{aligned}
J_1\ddot{\theta}_1 &= -k_{t1}(\theta_1-\theta_2) \\
J_2\ddot{\theta}_2 &= -k_{t1}(\theta_2-\theta_1)-k_{t2}(\theta_2-\theta_3) \qquad (4\text{-}48) \\
J_3\ddot{\theta}_3 &= -k_{t2}(\theta_3-\theta_2)
\end{aligned}
$$

Similar to Eq. (4-46), the frequency equation of the system is

$$\omega^2\left\{\omega^4-\left[k_{t1}\left(\frac{1}{J_1}+\frac{1}{J_2}\right)+k_{t2}\left(\frac{1}{J_2}+\frac{1}{J_3}\right)\right]\omega^2+k_{t1}k_{t2}\frac{J_1+J_2+J_3}{J_1J_2J_3}\right\}=0 \qquad (4\text{-}49)$$

The relative amplitudes at the principal modes can be obtained from Eq. (4-48) and expressed as

$$\frac{\Theta_1}{\Theta_2}=\frac{k_{t1}}{k_{t1}-\omega^2 J_1} \quad \text{and} \quad \frac{\Theta_2}{\Theta_3}=\frac{k_{t2}-\omega^2 J_3}{k_{t2}} \qquad (4\text{-}50)$$

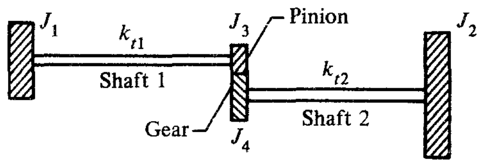
When $w^2=0$, the relative amplitudes of the disks are $\Theta_1/\Theta_2=\Theta_2/\Theta_3=1$. This indicates that the whole assembly may rotate as a rigid body. The relative amplitudes of the principal modes are shown in Fig. 4-14(*b*). Note that there is one sign change in the amplitudes for the first mode and two sign changes for the second mode.

Example 11. Geared systems

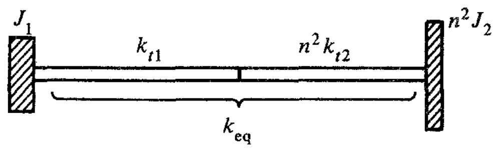
Let a two-shaft system in Fig. 4-15(*a*) be connected by a pair of gears. (a) Neglecting the inertial effect of the gears, determine the frequencies of the system. (b) Repeat part a but include the inertial effect of the gears.

Solution:

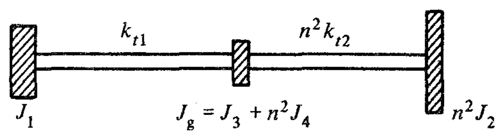
Since the two shafts are at different rotational speeds, it is expedient to find an equivalent system referring to a common shaft. Let N, be the number of



(a) Vibratory system



(*b*) Equivalent system, refer to shaft 1, neglect inertial effect of gears



(*c*) Equivalent system, refer to shaft I, including inertial effect of gears

FIG. 4-15. *Semidefinite geared system: Example 11.*

teeth on the pinion and N_2 that of the gear. Choosing shaft 1 as reference, the equivalent inertia of J_2 is $J_2^* = n^2 J_2$ as shown in Example 1, Chap. 3. Similarly, the equivalent spring constant of shaft 2 referring to shaft 1 is $k_{t2}^* = n^2 k_{t2}$.

(*a*) Referring to shaft 1 and neglecting the inertial effect of the gears, the equivalent system is shown in Fig. 4-15(*b*). The shafts are in series and the equivalent spring constant is

$$\frac{1}{k_{eq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}^*} \qquad \text{or} \qquad k_{eq} = \frac{n^2 k_{t1} k_{t2}}{k_{t1} + n^2 k_{t2}}$$

The natural frequencies from Eq. (4-46) are

$$f_n = \begin{cases} 0 \\ \dfrac{1}{2\pi}\sqrt{k_{eq}\left(\dfrac{1}{J_1} + \dfrac{1}{n^2 J_2}\right)} \end{cases}$$

(b) Including the gears, the system has four disks and therefore four equations of motion. J_3 of the pinion and J_4 of the gear can be combined to give $J_g = (J_3 + n^2 J_4)$, referring to shaft 1. Thus, the equivalent system consists of three disks and two shafts as shown in Fig. 4-15(*c*). The frequency equation would be identical to Eq. (4-49).

Note that the natural frequencies are the frequencies of oscillation of one disk relative to another, superposed on the rigid body rotation of the assembly. The natural frequencies can be calculated referring to one shaft or the other. The proof of this statement is left as an exercise.

4-8 FORCED VIBRATION—HARMONIC EXCITATION

The general form of the equations of motion of a two-degree-of-freedom system is shown in Eq. (4-4). If the excitation is harmonic, the equations can be solved readily by the impedance method developed in **Sec.** 2-6. The numerical solutions for systems with damping, however, are tedious. Computers can be used to alleviate the calculations.

Applying the impedance method to Eq. (4-4), we substitute the harmonic force vector $\{\mathbf{F}\}$ for the generalized force $\{Q(t)\}$, where $\{\mathbf{F}\} = \{\bar{F}e^{j\omega t}\} = \{\bar{F}\}e^{j\omega t}$ and the 2×1 matrix $\{\bar{F}\}$ is the phasor of $\{\mathbf{F}\}$. All the harmonic components in $\{\mathbf{F}\}$ are assumed of the same frequency ω. If not, one frequency can be treated at a time and the resultant response obtained by superposition. Let $\{\mathbf{X}\}$ be the harmonic response, where $\{\mathbf{X}\} \triangleq \{\bar{X}e^{j\omega t}\} = \{\bar{X}\}e^{j\omega t}$ and $\{\bar{X}\}$ is the phasor of $\{\mathbf{X}\}$. Applying the impedance method and factoring out $e^{j\omega t}$, Eq. (4-4) becomes

$$-\omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} + j\omega \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \end{bmatrix} \quad \textbf{(4-51)}$$

or

$$[K - \omega^2 M + j\omega C]\{\bar{X}\} = \{\bar{F}\} \quad \textbf{(4-52)}$$

where the matrices M, C, and K can be identified readily.

The equations above can be alternatively expressed as

$$\begin{bmatrix} k_{11} - \omega^2 m_{11} + j\omega c_{11} & k_{12} - \omega^2 m_{12} + j\omega c_{12} \\ k_{21} - \omega^2 m_{21} + j\omega c_{21} & k_{22} - \omega^2 m_{22} + j\omega c_{22} \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \end{bmatrix} \quad \textbf{(4-53)}$$

or

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \end{bmatrix} \quad \textbf{(4-54)}$$

or

$$Z(\omega)\{\bar{X}\} = \{\bar{F}\} \quad \textbf{(4-55)}$$

where

$$z_{ij} = (k_{ij} - \omega^2 m_{ij} + j\omega c_{ij}) \qquad \text{for} \qquad i, j = 1, 2 \quad \textbf{(4-56)}$$

and $Z(\omega)=[z_{ij}]$ is the impedance matrix. In other words, Eqs. (4-51) to (4-55) are different forms of the same equation, all of which can be summarized by Eq. (4-55).

The solution $\{\bar{X}\}$ gives the amplitude and phase angle of the response relative to the excitation $\{\bar{F}\}$. Premultiplying both sides of Eq. (4-55) by the inverse $Z(\omega)^{-1}$ of $Z(\omega)$ gives

$$\{\bar{X}\} = Z(\omega)^{-1}\{\bar{F}\} \tag{4-57}$$

For a two-degree-of-freedom system this can be written explicitly as

$$\bar{X}_1 = \frac{z_{22}\bar{F}_1 - z_{12}\bar{F}_2}{z_{11}z_{22} - z_{12}z_{21}} \quad \text{and} \quad \bar{X}_2 = \frac{-z_{21}\bar{F}_1 + z_{11}\bar{F}_2}{z_{11}z_{22} - z_{12}z_{21}} \tag{4-58}$$

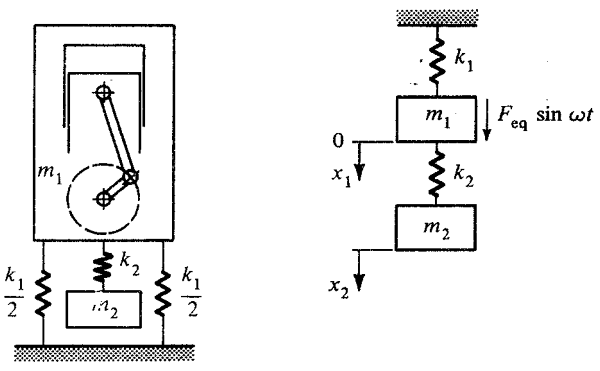
Equations (4-55) to (4-57) are equally applicable to n-degree-of-freedom systems. The elements of the impedance matrix $Z(\omega)$ in Eq. (4-55) become

$$z_{ij} = (k_{ij} - \omega^2 m_{ij} + j\omega c_{ij}) \quad \text{for} \quad i, j = 1, 2, \ldots, n \tag{4-59}$$

and $Z(\omega)$ is of order n. Note that each z_{ij} is identical in form to the mechanical impedance in Eq. (2-51) and $Z(\omega)$ is symmetric by the proper choice of coordinates as discussed in Sec. 4-4.

Example 12. Undamped dynamic absorber

Excessive vibration, due to near resonance conditions, is encountered in a constant speed machine shown in Fig. 4-16(*a*). The original system consists of m_1 and k_1. It is not feasible to change m_1 and k_1. *(a)* Show that a dynamic absorber, consisting of m_2 and k_2, will remedy the problem. (b) Plot the response curves of the system, assuming $m_2/m_1 = 0.3$. *(c)* Investigate the effect of the mass ratio m_2/m_1.



(a) Vibratory system (b) Equivalent system

FIG. 4-16. *Undamped dynamic* absorber: *Example 12.*

Solution:

The equations of motion for the equivalent system in Fig. 4-16(*b*) are

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) + F_{eq}\sin\omega t$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$

The impedance method can be applied directly, since the excitation is harmonic. From Eq. (4-53), we have

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix}\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} F_{eq} \\ 0 \end{bmatrix}$$

(*a*) Following Eq. (4-13), the frequency equation is obtained by equating the characteristic determinant $\Delta(\omega)$ of the coefficient matrix of $\{\bar{X}\}$ to zero, that is,

$$\Delta(\omega) = (k_1 + k_2 - \omega^2 m_1)(k_2 - \omega^2 m_2) - k_2^2 = 0 \qquad \textbf{(4-60)}$$

From Eq. 14-58), the phasors of the responses are

$$\bar{X}_1 = \frac{1}{\Delta(\omega)}(k_2 - \omega^2 m_2)F_{eq} \quad \text{and} \quad \bar{X}_2 = \frac{1}{\Delta(\omega)}k_2F_{eq} \qquad \textbf{(4-61)}$$

where $\Delta(\omega)$ is the characteristic determinant. Note that the amplitude $\bar{X}_1$ becomes zero at the excitation frequency $\omega = \sqrt{k_2/m_2}$. An undamped dynamic absorber is "tuned" for $k_1/m_1 = k_2/m_2$, such that $\bar{X}_1$ approaches zero at the resonance frequency of the original system.

(b) The frequency equation in Eq. *(4-60)* can be expressed as

$$\frac{m_1m_2}{k_1k_2}\omega^4 - \left[\left(1 + \frac{k_2}{k_1}\right)\frac{m_2}{k_2} + \frac{m_1}{k_1}\right]\omega^2 + 1 = 0$$

Since $k_2/k_1 = m_2/m_1$, a frequency ratio r is defined as $r = \omega/\sqrt{k_1/m_1} = \omega/\sqrt{k_2/m_2}$. *The* frequency equation reduces to

$$r^4 - (2 + m_2/m_1)r^2 + 1 = 0 \qquad \textbf{(4-62)}$$

From Eq. (4-61), the responses *can* be expressed as

$$\frac{\bar{X}_1}{F_{eq}/k_1} = \frac{1 - r^2}{r^4 - (2 + m_2/m_1)r^2 + 1}$$
$$\frac{\bar{X}_2}{F_{eq}/k_1} = \frac{1}{r^4 - (2 + m_2/m_1)r^2 + 1} \qquad \textbf{(4-63)}$$

The equations are plotted in Fig. *4-17* for $m_2/m_1 = 0.3$. The plus or minus sign of the amplitude ratio denotes that the response is either in-phase or 180° out-of-phase with the excitation. Resonances occur at $r = 0.762$ and *1.311*. Note that $x_1(t) = 0$ when $k_2 - 0^2m_2 = 0$. It can be shown from Eq. *(4-61)* that this condition occurs when the excitation $F_{eq}\sin\omega t$ is balanced by the spring force $-k_2x_2$.

(c) The frequency equation, **Eq. (4-62)**, is **plotted** in Fig. *4-18* to show the effect of the mass ratio m_2/m_1. When m_2/m_1 is small, the resonant

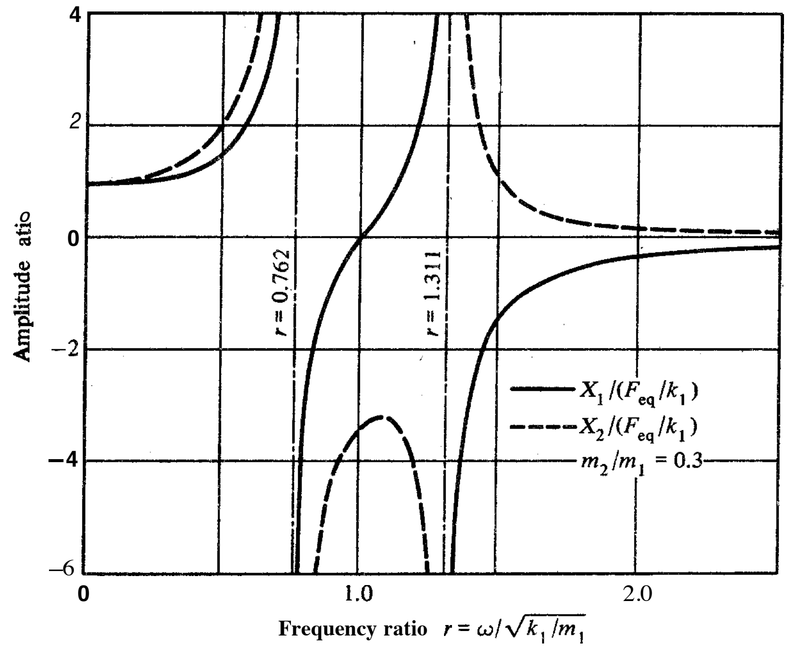


FIG. **4-17.** *Typical harmonic response of a two-degree-of-freedom system: Example 12.*

frequencies are close together about the resonance frequency of the original system. This means that there is little tolerance for variations in the excitation frequency, although $x_1(t)=0$ when $r=1$. Furthermore, it is observed in Eq. (4-63) that the amplitude X_2 of the absorber at $r=1$ can be large for small values of m_2/m_1. When m_2/m_1 is appreciable, the resonant frequencies are separated. For example, when $m_2/m_1=0.4$,

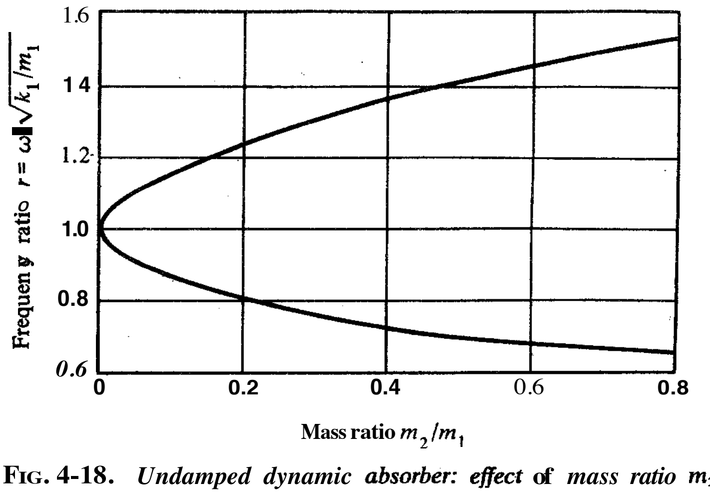


FIG. **4-18.** *Undamped dynamic absorber: effect of mass ratio m_2/m_1; Example 12.*

resonances occur at r equal to 0.73 and 1.37 times that of the original system. The amplitude X_2 of the absorber mass is correspondingly reduced at $r = 1$ for larger mass ratios.

Example 13. Dynamic *absorber with damping*

Consider the dynamic absorber in **Example** *12* in which a viscous damper c is installed in parallel with the spring k, as shown in Fig. **4-19(*b*)**. Briefly discuss the problem.

Solution:

From Eq. **(4-53)**, the equations of motion in phasor notations are

$$\begin{bmatrix} k_1+k_2-\omega^2 m_1+j\omega c & -k_2-j\omega c \\ -k_2-jwc & k_2-w^2m_2+j\omega c \end{bmatrix}\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}=\begin{bmatrix} F_{eq} \\ 0 \end{bmatrix}$$

From Eq. **(4-13)**, the corresponding frequency equation a

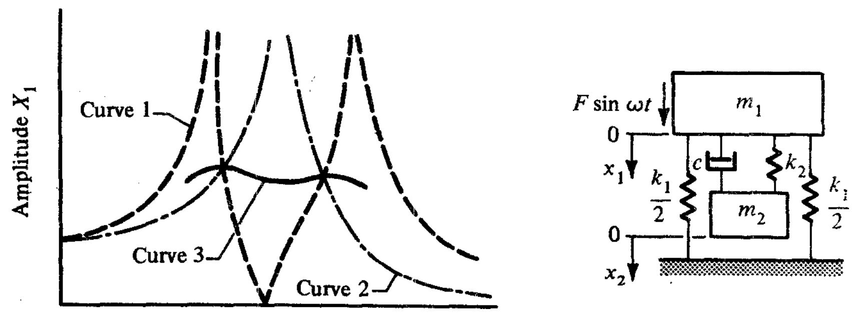
$$\Delta(\omega)=\begin{vmatrix} k_1+k_2-\omega^2 m_1+j\omega c & -k_2-j\omega c \\ -k_2-jwc & k_2-w^2m_2+j\omega c \end{vmatrix}=0$$

From Eq. (4-58), the phasors of the responses are

$$\bar{X}_1=\frac{1}{\Delta(\omega)}(k_2-\omega^2m_2+j\omega c)F_{eq} \quad \text{and} \quad \bar{X}_2=\frac{1}{\Delta(\omega)}(k_2+j\omega c)F_{eq}$$

where $\Delta(\omega)$ is the characteristic determinant. The values of $\bar{X}_1$ and $\bar{X}_2$ can be calculated **readily** using the programs in Chap. *9*.

The response curve of a properly tuned* dynamic absorber with appropriate damping is shown in Fig. **4-19(*a*)**. Curve 1 is that of an undamped system and curve 2 corresponding to $c=\infty$. Curve 3 of intermediate damping must pass through the intersections of these curves.



(*a*) Dynamic absorber with damping **(*b*) Vibratory system**

FIG. *4-19. Dynamic absorber with damping: Example 13.*

*** J. D. Den Hartog, *Mechanical Vibrations,* 4th ed., McGraw-Hill Book Company, New York, 1956, pp. 93–102. Note that the "tuned" condition of $k_1/m_1 = k_2/m_2$ in Example 12 is only for undamped absorbers. See Prob. 4-27 for discussion of dynamic absorbers with viscous damping.**

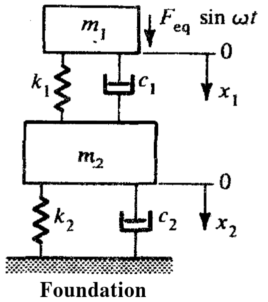


FIG. 4-20. *Vibration isolation: Example 14.*

Example 14. *Vibration* ***isolation***

A constant speed machine is isolated as shown in Fig. 3-23(a) and the girls in the office complain of the annoying vibration transmitted from the machine. It is proposed (1)to mount the machine m_1 on a cement block m_2 as shown in Fig. 4-20, or (2) to bolt m_1 rigidly to m_2. Assume $m_2/m_1 = 4$, $k_1 = k_2$, and the excitation frequency $\omega = 2\omega_{n1} = 2\sqrt{k_1/m_1}$. *(a)* **Find** the magnitudes of $x_1(t)$ and $x_2(t)$ and the force F_T transmitted to the rigid foundation. (b) Neglecting the damping in the system for the estimation, would you approve proposal 1 or 2?

Solution:

(a) From Eq. (4-53), the equations of motion in phasor notations are

$$\begin{bmatrix} k_1 - \omega^2 m_1 + j\omega c_1 & -k_1 - j\omega c_1 \\ -k_1 - j\omega c_1 & k_1 + k_2 - \omega^2 m_2 + j\omega(c_1 + c_2) \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} F_{eq} \\ 0 \end{bmatrix}$$

The characteristic determinant $\Delta(\omega)$ can be identified from the equation above. Equating $\Delta(\omega)$ to zero gives the frequency equation. Thus,

$$\Delta(\omega) = \begin{vmatrix} k_1 - \omega^2 m_1 + j\omega c_1 & -k_1 - j\omega c_1 \\ -k_1 - j\omega c_1 & k_1 + k_2 - \omega^2 m_2 - j\omega(c_1 + c_2) \end{vmatrix} = 0$$

From Eq. (4-58), the response of the system is

$$\bar{X}_1 = \frac{1}{\Delta(\omega)}[k_1 + k_2 - \omega^2 m_2 + j\omega(c_1 + c_2)]F_{eq}$$

$$\bar{X}_2 = \frac{1}{\Delta(\omega)}(k_1 + j\omega c_1)F_{eq}$$

The force $\bar{F}_T$ is transmitted to the foundation through the **spring** k_2 and the damper c_2. Thus,

$$\bar{F}_T = \bar{X}_2(k_2 + j\omega c_2) = \frac{1}{\Delta(\omega)}(k_1 + j\omega c_1)(k_2 + j\omega c_2)F_{eq}$$

The solution $\{\bar{X}\}$ and the force transmitted $\bar{F}_T$ can be calculated.from the equations above.

(b) Proposal 1: The frequency equation of the undamped system is

$$\Delta(\omega) = \begin{vmatrix} k_1 - \omega^2 m_1 & -k_1 \\ -k_1 & k_1 + k_2 - \omega^2 m_2 \end{vmatrix} = 0$$

Substituting the given conditions $m_2/m_1 = 4$, etc., the frequency equation can be expressed as

$$\Delta(\omega) = k_1 k_2 (1 - 6r^2 + 4r^4) = 0$$

where $r = \omega/\omega_{n1} = \omega/\sqrt{k_1/m_1}$. Correspondingly, we get

$$\bar{X}_1 = \frac{1}{\Delta(\omega)}(k_1 + k_2 - \omega^2 m_2)F_{eq} \qquad \text{or} \qquad \frac{X_1}{F_{eq}/k_1} = \frac{2 - 4r^2}{1 - 6r^2 + 4r^4}$$

$$\bar{X}_2 = \frac{1}{\Delta(\omega)} k_1 F_{eq} \qquad \text{or} \qquad \frac{\bar{X}_2}{F_{eq}/k_1} = \frac{1}{1 - 6r^2 + 4r^4}$$

$$\bar{F}_T = \frac{1}{\Delta(\omega)} k_1 k_2 F_{eq} \qquad \text{or} \qquad \frac{\bar{F}_T}{F_{eq}} = \frac{1}{1 - 6r^2 + 4r^4}$$

For the given values, resonance occurs at $r = 0.437$ and 1.14. At $\omega = 2\omega_{n1}$ or $r = 2$, we have $X_1 = |\bar{X}_1| = 0.341 F_{eq}/k_1$, $X_2 = |\bar{X}_2| = 0.024 F_{eq}/k_1$, and $F_T = |\bar{F}_T| = 0.024 F_{eq}$.

Proposal 2: If m_1 is attached to m_2, the system has one degree of freedom. The equation of motion in phasor notations is

$$[k_2 - 0^2(m_1 + m_2)]\bar{X}_2 = F_{eq}$$

Using the given data and normalizing the results by k_1, we get

$$\frac{\bar{X}_2}{F_{eq}/k_1} = \frac{1}{1 - 5r^2}$$

$$\bar{F}_T = k_2 \bar{X}_2 \qquad \text{and} \qquad \frac{\bar{F}_T}{F_{eq}} = \frac{1}{1 - 5r^2}$$

Resonance occurs at $r = 0.45$. At the excitation frequency $\omega = 2\omega_{n1}$ or $r = 2$, $X_2 = |\bar{X}_2| = 0.053 F_{eq}/k_1$ and $F_T = |\bar{F}_T| = 0.053 F_{eq}$. Hence the force transmitted is higher than that in proposal **1.**

4-9 INFLUENCE COEFFICIENTS

The *method of influence coefficients* gives an alternative procedure to formulate the equations of motion of a dynamic system. It is widely used in the analysis of structures, such as an aircraft. A spring can be described by its stiffness or its *compliance,* which is synonymous to the flexibility influence coefficient. We shall first (1) show Maxwell's reciprocity theorem, (2) relate the stiffness and flexibility matrices, and then (3) illustrate the method of influence coefficients.

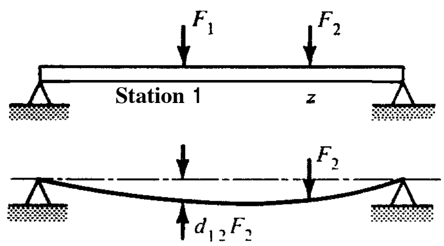


FIG. 4-21. Method of influence coefficients.

An influence ***coefficient*** d_{ij} defines the static elastic property of a system. The quantity d_{ij} is the deflection at station i owing to a unit force applied at station *j*, when this force is the only force applied. Consider the beam shown in Fig. 4-21. The vertical force F_1 is applied at station 1 and F_2 at station 2.

First, let F_1 be applied to station 1 and then F_2 to station 2. When F_1 is applied alone, the deflection at station 1 is F_1d_{11}. The potential energy in the beam, by virtue of its deflection, is $\frac{1}{2}F_1^2d_{11}$. Now, when F_2 is applied, the additional deflection at station 1 due to F_2 is F_2d_{12}. The work by F_1 corresponding to this deflection is $F_1(F_2d_{12})$. Thus, the total potential energy U of the system due to F_1 and F_2 is

$$U=\tfrac{1}{2}F_1^2d_{11}+F_1(F_2d_{12})+\tfrac{1}{2}F_2^2d_{22}$$

Secondly, let F_2 be applied to station 2 and then F_1 to station *1*. It can be shown that the potential energy U due to F_2 and F_1 is

$$U=\tfrac{1}{2}F_2^2d_{22}+F_2(F_1d_{21})+\tfrac{1}{2}F_1^2d_{11}$$

The potential energies for the two methods of loading must be the same, since the final states of 'the system are identical. Comparing the expressions for U, we deduce that $d_{12}=d_{21}$ for the system with two loads. This is called Maxwell's reciprocity theorem.

For the general case, we have

$$d_{ij}=d_{ji} \qquad \text{for} \qquad i, j=1,2,\ldots n \tag{4-64}$$

which holds for all linear systems. When the force F is generalized to represent a force or a moment, the influence coefficient d_{ij} correspondingly represents a rectilinear or an angular displacement. Furthermore, when the deflections due to the inertia forces are considered, we obtain the equations of motion of the system.

During vibration, the inertia force associated with each mass is transmitted throughout the system to cause a motion at each of the other masses. For the undamped free vibration of a two-degree-of-freedom system, we have

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} -m_1\ddot{q}_1 \\ -m_2\ddot{q}_2 \end{bmatrix} \tag{4-65}$$

or

$$\{q\} = [d_{ij}]\{-m\ddot{q}\} \tag{4-66}$$

where {q} is a generalized displacement vector, $\{-m\ddot{q}\}$ the generalized force vector of the inertia forces, and $[d_{ij}]$ is the flexibility (influence coefficient) matrix. For example, the deflection q_1 at station I is due to the combined effect of the inertia forces $-m_1\ddot{q}_1$ and $-m_2\ddot{q}_2$. The total deflection q_1 is $[d_{11}(-m_1\ddot{q}_1)+d_{12}(-m_2\ddot{q}_2)]$.

From **Eq. (4-26),** in the absence of dynamic coupling, the equations of motion for the free vibration of an undamped system can be expressed as

$$\{m\ddot{q}\} = -[k_{ij}]\{q\} \tag{4-67}$$

where $[k_{ij}]$ is the stiffness matrix. **Premultiplying** Eq. (4-66) by the inverse $[d_{ij}]^{-1}$ of $[d_{ij}]$ and rearranging, we get

$$\{m\ddot{q}\} = -[d_{ij}]^{-1}\{q\}$$

Comparing the last two equations, it is evident that

$$[k_{ij}] = [d_{ij}]^{-1} \qquad \text{or} \qquad [k_{ij}][d_{ij}] = I \tag{4-68}$$

where **I** is a unit matrix. This is to say that $[k_{ij}]$ is the inverse of $[d_{ij}]$ and vice versa.

Example 15

Write the equations of motion for the system shown in Fig. 4-2(*a*) by the **method** of influence **coefficients** and find the frequency equation.

Solution:

To find the influence coefficients, let a unit static force be applied to m,. The springs k and k_2 are in series and their combination is in parallel with k_1. Thus,

$$k_{eq1} = k_1 + \frac{kk_2}{k+k_2}$$

The **corresponding** deflection of m_1 is d_{11}.

$$d_{11} = \frac{1}{k_{eq1}} = \frac{k+k_2}{k_1k_2 + k(k_1+k_2)}$$

Since the deflection of a spring is inversely proportional to its stiffness and the deflection of m_1 is d_{11}, it can be shown that the corresponding deflection

d_{21} of m_2 is

$$d_{21}=\frac{k}{k+k_2}d_{11}=\frac{k}{k_1k_2+k(k_1+k_2)}=d_{12}$$

Similarly, considering a unit static force at m_2, we get

$$d_{22}=\frac{k+k_1}{k_1k_2+k(k_1+k_2)}$$

Combining the influence coefficients yields

$$[d_{ij}]=\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}=\frac{1}{k_1k_2+k(k_1+k_2)}\begin{bmatrix} k+k_2 & k \\ k & k+k_1 \end{bmatrix}$$

The equations of motion from Eq. (4-66) are

$$\{x\}=[d_{ij}]\{-m\ddot{x}\} \tag{4-69}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\frac{1}{k_1k_2+k(k_1+k_2)}\begin{bmatrix} k+k_2 & k \\ k & k+k_1 \end{bmatrix}\begin{bmatrix} -m_1\ddot{x}_1 \\ -m_2\ddot{x}_2 \end{bmatrix} \tag{4-70}$$

Note that the stiffness matrix from Eq. (4-9) is

$$[k_{ij}]=\begin{bmatrix} k+k_1 & -k \\ -k & k+k_2 \end{bmatrix}$$

It can be shown readily that $[k_{ij}]=[d_{ij}]^{-1}$. Moreover, the premultiplication of Eq. (4-70) by $[d_{ij}]^{-1}$ will give Eq. (4-9), which are the equations of motion of the same system by Newton's second law.

To find the frequency equation, we substitute $(j\omega)^2$ for the second time derivative and express Eq. (4-69) in phasor notation as

$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}=\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\begin{bmatrix} \omega^2 m_1\bar{X}_1 \\ \omega^2 m_2\bar{X}_2 \end{bmatrix}$$

or

$$\begin{bmatrix} 1-d_{11}\omega^2 m_1 & -d_{12}\omega^2 m_2 \\ -d_{21}\omega^2 m_1 & 1-d_{22}\omega^2 m_2 \end{bmatrix}\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4-71}$$

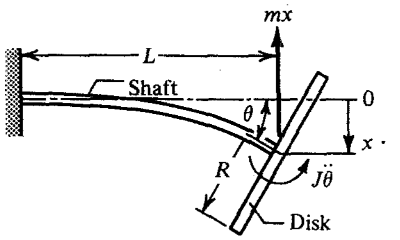
The frequency equation is obtained by equating the determinant $\Delta(\omega)$ of the coefficient matrix of $\{\bar{X}\}$ to zero.

$$\Delta(\omega)=\begin{vmatrix} 1-d_{11}\omega^2 m_1 & -d_{12}\omega^2 m_2 \\ -d_{21}\omega^2 m_1 & 1-d_{22}\omega^2 m_2 \end{vmatrix}=0 \tag{4-72}$$

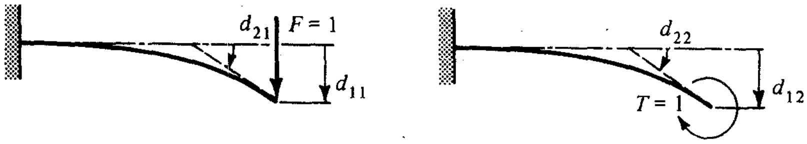
Substituting the values for d_{ij}, expanding the determinant $\Delta(\omega)$, and simplifying, the frequency equation becomes

$$m_1m_2\omega^4-[(k+k_1)m_2+(k+k_2)m_1]\omega^2+[k_1k_2+k(k_1+k_2)]=0$$

This is identical to the frequency equation in Eq. (4-14) by Newton's second law for the same system.



(a) **Vibratory system**



(*b*) Determination of influence coefficients

FIG. **4-22.** ***Influence coefficients due to*** **force** ***and moment; Example 16.***

Example *16*

Determine the natural frequency of the system shown in Fig. **4-22(*a*).** Assume **that (1)** the flexural stiffness of the shaft is ***EI***, (2) the inertia effect of the shaft is negligible, (3) the shaft is horizontal in its static equilibrium position, and **(4)** the mass moment of inertia of the disk is $J = mR^2/4$ where $R = L/4$.

Solution:

The inertia forces are as shown in Fig. **4-22(*a*)** and the influence coefficients are defined in Fig. **4-22(*b*).** From elementary beam theory, it can be shown that the **influence** coefficients are

$$d_{11} = L^3/3EI, \qquad d_{22} = L/EI, \qquad d_{12} = d_{21} = L^2/2EZ$$

The equations of motion from Eq. **(4-65)** are

$$\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} -m\ddot{x} \\ -J\ddot{\theta} \end{bmatrix}$$

or

$$\begin{bmatrix} x \\ \theta \end{bmatrix} = -\frac{L}{6EI} \begin{bmatrix} 2L^2 & 3L \\ 3L & 6 \end{bmatrix} \begin{bmatrix} m\ddot{x} \\ J\ddot{\theta} \end{bmatrix}$$

Following the last example, this can be expressed in phasor notations as shown in Eq. (4-71).

$$\begin{bmatrix} 6EI - 2\omega^2 mL^3 & -3\omega^2 JL^2 \\ -3\omega^2 mL^2 & 6EI - 6\omega^2 JL \end{bmatrix} \begin{bmatrix} \bar{X} \\ \bar{\Theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The frequency is obtained by equating the determinant $\Delta(\omega)$ of the coefficients of $\{\bar{X} \quad \bar{\Theta}\}$ to zero.

$$\Delta(\omega) = \begin{vmatrix} 6EI - 2\omega^2 mL^3 & -3\omega^2 JL^2 \\ -3\omega^2 mL^2 & 6EI - 6\omega^2 JL \end{vmatrix} = 0$$

Substituting $J = mR^2/4$ and $R = L/4$, and expanding $\Delta(\omega)$, we get

$$\omega^4 - 268(EI/mL^3)\omega^2 + 768(EI/mL^3)^2 = 0$$

Hence

$$w = 1.70\sqrt{EI/mL^3} \qquad \text{and} \qquad 16.3\sqrt{EI/mL^3}$$

4-10 SUMMARY

The chapter introduces the theory of discrete systems from the generalization of a two-degree-of-freedom system shown in Fig. 4-1. The equations of motion in Eq. (4-1) through (4-4) are coupled, because the equation for one mass is influenced by the motion of the other mass of the system.

The modes of vibration are examined in **Sec.** 4-3 for undamped free vibrations. The natural frequencies are obtained from the characteristic equation in Eq. (4-13). A mode of vibration, called the principal mode, is associated with each natural frequency. At a principal mode, (1) the entire system executes synchronous harmonic motion at a natural frequency and (2) the relative amplitudes of the masses are constant, as shown in Eq. (4-16) and illustrated in Fig. 4-2(*b*). The relative amplitudes define the modal vector for the given mode. The general motion is the superposition of the modes, as shown in Eq. (4-17).

A system can he described by more than one set of generalized coordinates $\{q\}$. In Eqs. (4-27) through (4-29), it is shown that the elements of the mass matrix and the stiffness matrix as well as the type of coupling in **the** equations of motion are dependent on the coordinates **selected** for **the** system description. Hence coordinate coupling is not an **inherent** property of the system. The coordinates that uncouple the equations are called the principal coordinates $\{p\}$. The coordinates {p) and $\{q\}$ are related by the modal matrix [u] as shown in Eq. (4-34).

A method for finding the modal matrix is shown in See. 4-6. The equations of motion can be uncoupled by means of the modal matrix. Thus, each uncoupled equation can be treated as an independent **one-**degree-of-freedom system. The results can be expressed in the {p) or $\{q\}$ coordinates as desired. The technique is conceptually simple, but computers are necessary for the numerical solutions.

Many practical problems can be represented as semidefinite systems as discussed in **Sec.** 4-7. A system is **semidefinite** if it can move as a rigid body. Correspondingly, at least one of its natural frequencies is zero.

The harmonic response of discrete systems can be found readily by the mechanical impedance method. Using phasor notations, the equations of motion can be expressed as $Z(\omega)\{\bar{X}\} = \{\bar{F}\}$ in Eq. *(4-55)* and the response as $\{\bar{X}\} = Z(\omega)^{-1}\{\bar{F}\}$ in Eq. *(4-57)*.

The method of influence coefficients in Sec. 4-9 gives an alternative procedure to formulate the equations of motion. From Maxwell's reciprocity theorem, the flexibility matrix $[d_{ij}]$ is symmetric. The inverse $[d_{ij}]^{-1}$ of the flexibility matrix is the stiffness matrix $[k_{ij}]$. Thus, except for the technique in obtaining the equations of motion and certain advantages in its application, the concepts of vibration in the previous sections can be applied readily in this method.

PROBLEMS

Assume all the systems in the figures to follow are shown in their static equilibrium positions.

4-1 Consider the system in Fig. 4-2(*a*). Let $m_1 = m_2 = 10$ kg, $k_1 = k_2 = 40$ N/m, and $k = 60$ N/m. *(a)* Write the equations of motion and the frequency equation. (b) Find the natural frequencies, the principal modes, and the modal matrix. (c) Assume $\{x(0)\} = \{1 \quad 0\}$ and $\{\dot{x}(0)\} = \{0 \quad 1\}$. Plot $x_1(t)$ and $x_2(t)$ and their harmonic components. (d) Assume $\{x(0)\} = \{0 \quad 0\}$ and $\{\dot{x}(0)\} = \{1 \quad -1\}$. Find $x_1(t)$ and $x_2(t)$.

4-2 Repeat Prob. *4-1* if $m_1 = m_2 = 10$ kg, $k_1 = 40$ N/m, $k_2 = 140$ N/m, and $k =$ 6b N/m. Are the motions $\{x(t)\}$ periodic?

4-3 A *200-kg* uniform bar is supported by springs at the ends as illustrated in Fig. *4-5*. The total length is $L = 1.5$ m, $k_1 = 18$ kN/m, and $k_2 = 22$ kN/m. (a) Write the equations of motion and the frequency equation. (b) Find the natural frequencies, the principal modes, and the modal matrix. (c) If $x(0) = 1$, $\dot{x}(0) = \theta(0) = \dot{\theta}(0) = 0$, find the motions $x(t)$ and $\theta(t)$. *(d)* Illustrate the principal modes, such as shown in Fig. 4-6.

4-4 For the three-degree-of-freedom system in Fig. *4-7*, if $J_1 = 2J_2$, $J_2 = 2J_3$, $k_{t1} = 2k_{t2}$, and $k_{t2} = 2k_{t3}$, find the motions $\theta_1(t)$, $\theta_2(t)$, and $\theta_3(t)$.

4-5 For each of the systems shown in Fig. *P4-1*, specify the coordinates to describe the system, write the equations of motion, and find the frequency equation.

(a) A double pendulum.

(b) The arm is horizontal in its static equilibrium position.

(c) Three identical pendulums.

(d) A double compound pendulum.

(e) A schematic representation of an overhead crane.

(f) The system is constrained to move in the plane of the paper.

(g) The bar and the shaft are initially horizontal. The shaft deflects vertically. The bar moves vertically as well as rotates in a vertical plane.

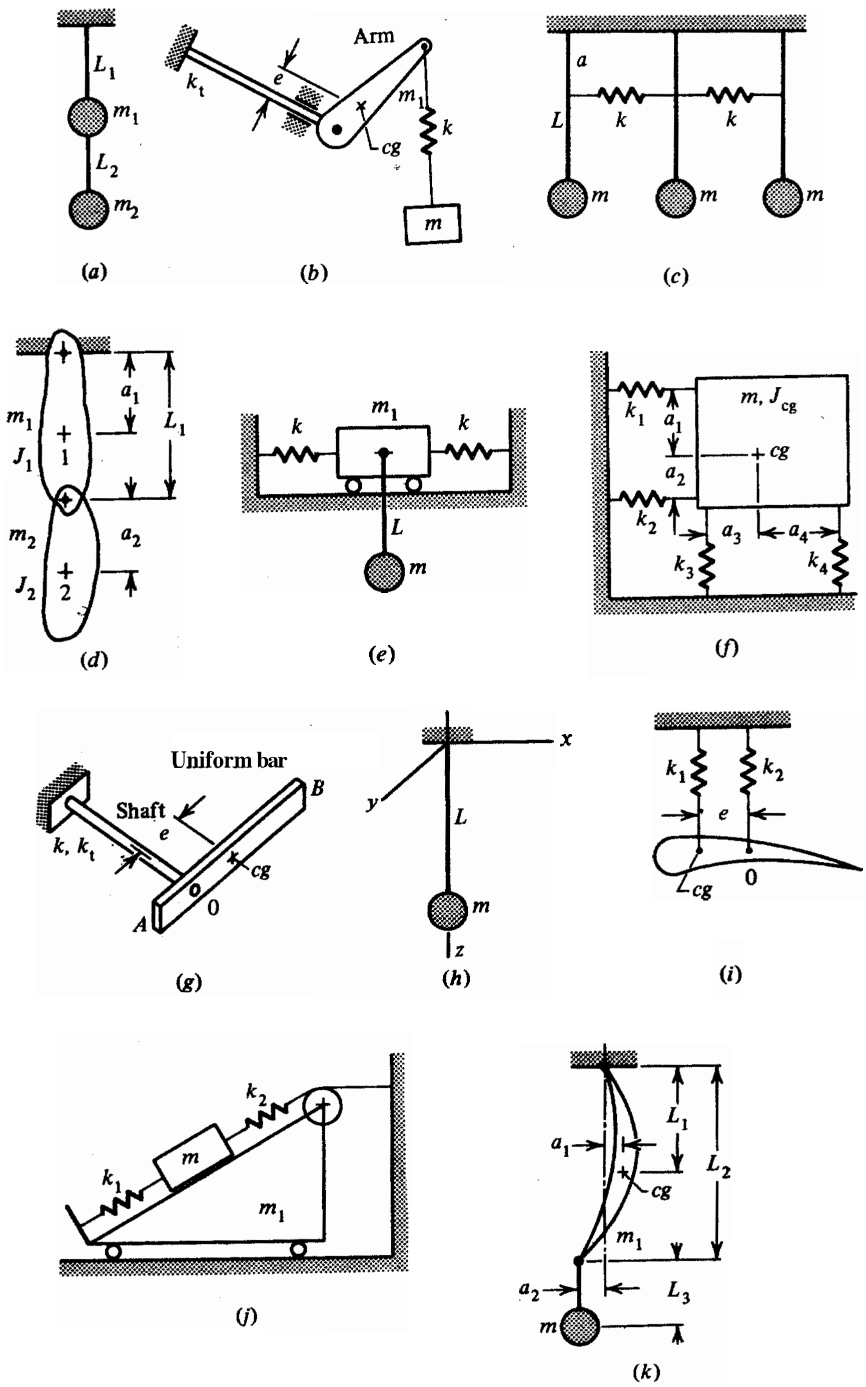


FIG. P4-1. *Vibratory systems.*

(h) A spherical pendulum.

(i) The airfoil moves vertically and pivots about cg.

(j) Assume that there is no friction between m and m_1.

(k) The pendulums are constrained to move in the plane of the paper.

4-6 For the double pendulum in Fig. P4-1(*a*), let $m_1 = m_2$ and $L_1 = L_2$. (*a*) If x_1 is the horizontal displacement of m_1 and x_2 that of m_2, write the equations of motion in terms of (x_1, x_2) and find the natural frequencies. (b) If θ_1 and θ_2 are the angular displacements of the pendulums, write the equations of motion in terms of (θ_1, θ_2) and find the natural frequeneies.

4-7 Referring to Fig. *4-9* on coordinate coupling, (*a*) convert Eq. *(4-28)* to (4-27), using the relations $x_2 = x_1 - e\theta$, $L_3 = L_1 - e$, $L_4 = L_2 + e$, and $J_2 = J_1 + me^2$; and (b) convert Eq. *(4-29)* to (4-27), using the relations $x_3 = x_1 - L_1\theta$, $L = L_1 + L_2$, and $J_3 = J_1 + mL_1^2$.

4-8 Referring to Fig. P4-1(*g*), write the equations of motion of the system if the vertical displacement of the bar is measured from: (*a*) the mass center cg; (b) the point 0; (c) the point A; (d) the point B.

4-9 Show that the frequency equation for the case of non-symmetrical matrices in Example 6 is identical to Eq. (4-14).

4-10 A company crates its products for shipping as shown in Fig. P4-2(*a*). The skid is securely mounted on a truck. Experience indicates that this method of crating is satisfactory. To cut the shipping coat, it is proposed to put two items in a crate as shown in Fig. P4-2(*b*). Would you approve this proposal?

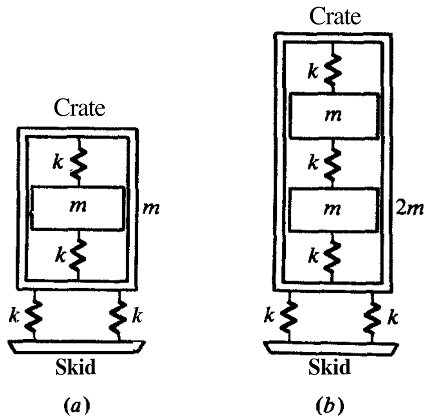


FIG. *P4-2.*

4-11 Consider an undamped three-degree-of-freedom system

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix}$$

where $\{F(t)\}$ is a vector of transient excitations. (a) Find the frequency equation and the natural frequencies. (b) Determine the modal vectors and the modal matrix. (c) Verify that the modal vectors are orthogonal relative to the matrices M and K as shown in Eq. (4-40). (d) Write the uncoupled equations as indicated in Eq. (4-41).

4-12 Repeat Prob. 4-11 for the equations

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix}$$

4-13 Determine the motions $\theta_1(t)$ and $\theta_2(t)$ of the semidefinite system in Fig. 4-13 for the initial conditions: (a) $\{\theta(0)\} = \{\theta_{10} \quad \theta_{20}\}$ and $\{\dot{\theta}(0)\} = \{0 \quad 0\}$; (b) $\{\theta(0)\} = \{0 \quad 0\}$ and $\{\dot{\theta}(0)\} = \{0 \quad \theta_{20}\}$.

4-14 For the semidefinite system shown in Fig. 4-14, if $J_1 = 1.2\ \text{m}^2\cdot\text{kg}$, $J_2 = J_3 = 2J_1$, $k_{t1} = 25 \times 10^3\ \text{N}\cdot\text{m/rad}$, and $k_{t2} = 2k_{t1}$, find the natural frequencies and the relative amplitudes at the principal modes.

4-15 For the system in Prob. 4-14, find the motions $\{\theta(t)\}$ if the initial conditions are: (a) $\{\theta(0)\} = (0.1 \quad 0 \quad 0\}$ and $\{\dot{\theta}(0)\} = (0)$; (b) $\{\theta(0)\} = \{0\}$ and $\{\dot{\theta}(0)\} = \{10 \quad 0 \quad 0\}$.

4-16 Neglecting the inertial effect of the pinion and the gear in Fig. 4-15, let $J_1 = 0.2\ \text{m}^2\cdot\text{kg}$, $J_2 = 4J_1$, $k_{t1} = 60 \times 10^3\ \text{N}\cdot\text{m/rad}$, $k_{t2} = 7k_{t1}$, and the gear ratio $= 3:1$. Find the natural frequencies of the system: (a) referring to shaft 1; (b) referring to shaft 2.

4-17 Assume a variable speed engine with four impulses per revolution is attached to J_1 of the gear system described in Prob. 4-16. Find the resonance speed of the gear system. What would be the resonance speed if the engine is attached to J_2?

4-18 Assume the inertial effect of the pinion and gear in Prob. 4-16 is not negligible. Repeat Prob. 4-16 if $J_3 = 0.02\ \text{m}^2\cdot\text{kg}$ and $J_4 = 20J_3$.

4-19 Find the motions $x_1(t)$ and $x_2(t)$ of the semidefinite system shown in Fig. P4-3(a), where $\delta(t)$ is a unit impulse. Assume zero initial conditions.

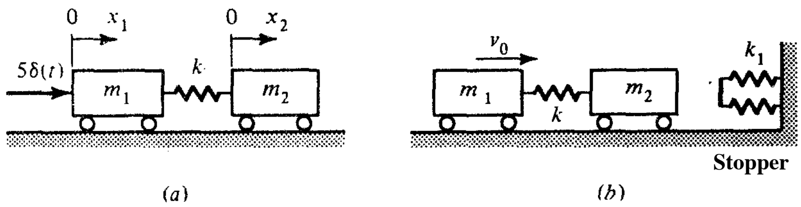


FIG. P4-3.

4-20 A semidefinite system strikes a stopper as shown in Fig. P4-3(b). Find the maximum force transmitted to the base of the stopper. Assume the velocity v_0 is constant and the springs are initially unstressed. Assume $m_1 = m_2$ and $k_1 = 2k$.

4-21 A branched-geared system is shown in Fig. P4-4. Assume the inertial effect of the shafts and the coupling is negligible. The gear ratio of the gears $J_b : J_c = 1:2$ and $J_b : J_d = 1:3$. The data as shown are in the SI units. (a)

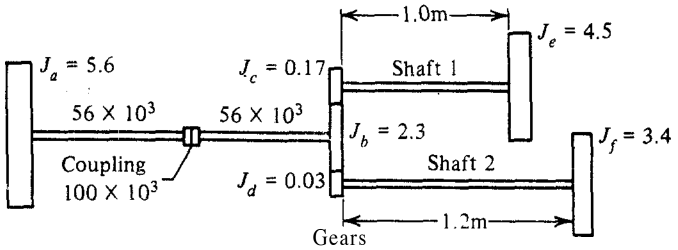
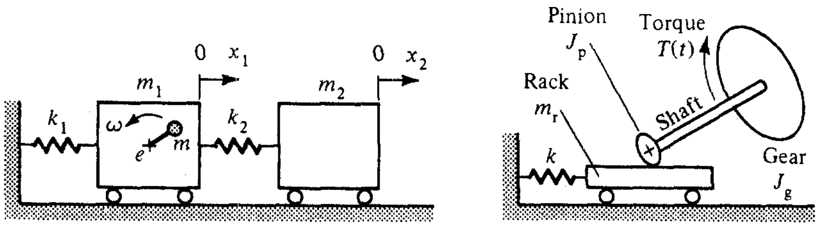


FIG. **P4-4.** *Branched-geared system.*

Specify the diameters of the shafts **1** and **2** such that the system has only two numerically distinct nonzero natural frequencies. (b) Find the natural frequencies.

4-22 Assuming harmonic excitations, find the steady-state response of each of the systems in Fig. **P4-5.**

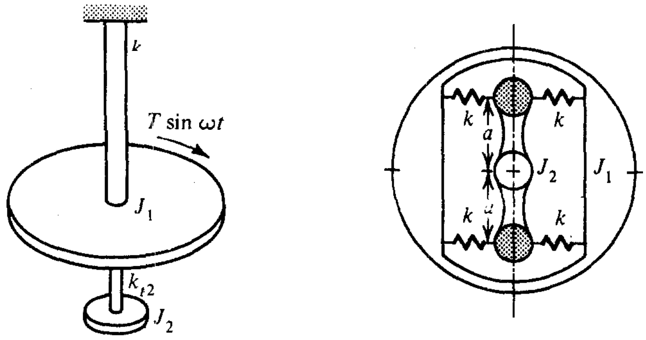


(*a*) (*b*)

FIG. P4-5.

4-23 An air compressor of **270** kg mass is mounted as shown in Fig. **4-16.** The normal operating speed is **1,750** rpm. **(a)** If the resonant frequencies should be at least ±**20** percent from the operating speed, specify k_1, k_2, and m_2. **(b)** What is the amplitude of m_2 at the operating speed?

4-24 A torque T sin ωt is applied to J_1 of the torsional system in Fig. **P4-6(*a*).** If $J_1 = 0.5\ m^2 \cdot kg$, $k_{t1} = 560 \times 10^3$ **m·N/rad,** $T = 226$ **m·N,** and $\omega = 10^3$ **rad/s,** specify J_2 and k_{t2} of the absorber such that the resonant frequencies are **20** percent from the excitation frequency.



(*a*) (*b*)

FIG. P4-6.

4-25 Repeat Prob. **4-24** if the absorber is as shown in Fig. **P4-6(*b*).**

4-26 A horizontal force $F \sin \omega t$ is applied to the mass m_1 of the system shown in **Fig. P4-1(*e*).** Find the condition for which m_1 is stationary.

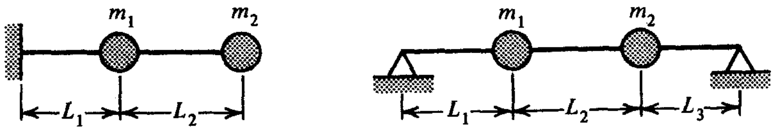
4-27 A dynamic absorber with damping is shown in Fig. **4-19.** For optimum design, the amplitudes of X_1 are equal at the **intersections** of curves **1** and **2.** Show that the relation $k_2/k_1 = m_1 m_2/(m_1 + m_2)^2$ is satisfied for this optimum.

4-28 Find the influence coefficients for the system shown in Fig. **4-5.** Write the dynamic equations. Show that the frequency equation can be reduced to that obtained from Newton's second law.

4-29. Repeat Prob. **4-28** for the system shown in Fig. **4-7.** Assume $J_1 = J_2 = J_3$ and $k_{t1} = k_{t2} = k_{t3}$.

4-30 **Find** the influence wefficients and the frequency equations for each of the systems shown in Fig. **P4-1(*a*)** to **(*c*).**

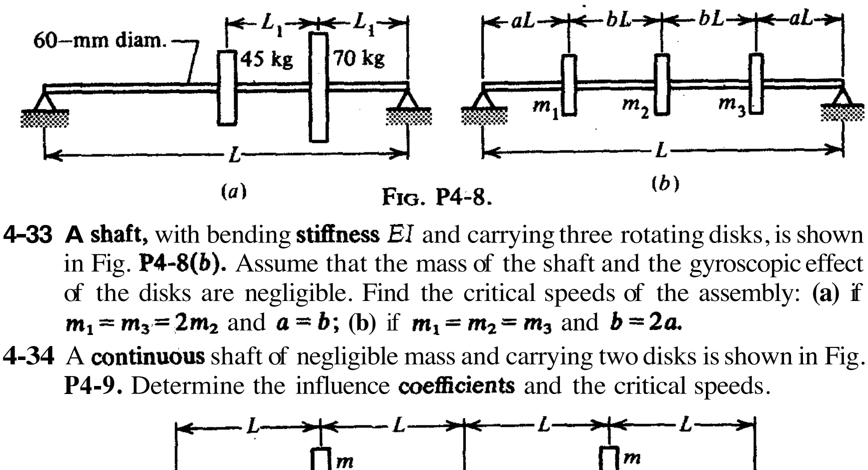
4-31 Find the influence coefficients for each of the systems shown in Fig. **P4-7.** Assume that the beams are of negligible mass.



(a) Uniform cantilever (b) Simply supported beam

FIG. **P4-7.**

4-32 A shaft carrying two rotating disks is shown in **Fig. P4-8(*a*).** Find the influence coefficients and the critical speeds of the assembly. Assume that (1) the deflections of the bearings and the gyroscopic effect of the disks are negligible, and (2) $L_1 = 150$ mm and $L = 600$ mm.



(a) **FIG. P4-8.** *(b)*

4-33 A shaft, with bending **stiffness** EI and carrying three rotating disks, is shown in Fig. **P4-8(*b*).** Assume that the mass of the shaft and the gyroscopic effect of the disks are negligible. Find the critical speeds of the assembly: **(a)** if $m_1 = m_3 = 2m_2$ and $a = b$; (b) if $m_1 = m_2 = m_3$ and $b = 2a$.

4-34 A **continuous** shaft of negligible mass and carrying two disks is shown in Fig. **P4-9.** Determine the influence **coefficients** and the critical speeds.

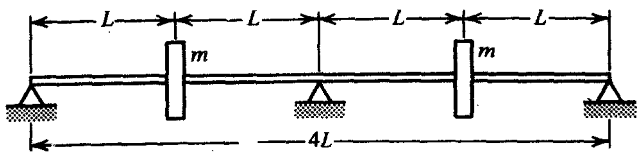


FIG. P4-9.

Computer problems:

Remarks: Transient response and frequency response are examined in the problems. The modal analysis for the transient response of positive-definite undamped systems with distinct frequencies can be examined by parts or by a combined program. Problems **4-35** to **4-39** show the parts of the modal analysis, following the theory developed in the chapter. Using the program TRESPUND, Probs. **4-40** to **4-42** show the combined program, which is a collection of subroutines. The remaining problems deal with the **frequency** response of discrete systems.

4-35 *Characteristic equation.* Use the program COEFF listed in Fig. **4-11** to find the coefficients of the characteristic equation of each of the following systems:

(a) $$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}+\begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}+\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

4-36 *Natural frequencies.* The roots ω^2 of the characteristic equation in Eq. **(4-38)** yield the natural frequencies ω of the system. Write a program to find the roots of the characteristic equation in Prob. **4-35.** *Hint:* Use the subroutine $ROOT in Fig. **C-7.**

4-37 *Modal matrix.* The steps to obtain a modal matrix are: (1) substituting an eigenvalue **A(=w²)** in Eq. **(4-37)**, $[-\omega^2M+K]\{q\}=\{0\}$; **(2)** finding the solutions of the homogeneous equations, as shown in Example **9,** to obtain a modal vector; **(3)** combining the modal vectors to form a modal matrix. For the data in Prob. **4-35,** write a program to

- **(a)** convert the equations of motion $M\{\ddot{x}\}+K\{x\}=\{0\}$ to the form $\{\ddot{x}\}+H\{x\}=\{0\}$, where $H=M^{-1}K$ is *a* dynamic matrix,
- **(b)** use the subroutine $ROOT in Fig. **C-7** to find the eigenvalues,
- (c) use the subroutine **$HOMO** in Fig. **C-9** to solve the homogeneous equations in order to find the modal vectors, and
- (**d**) combine the modal vectors to obtain the modal matrix.

4-38 *Modal matrix.* Repeat Prob. *4-37* by using the subroutine **$MODL** listed in Fig. *C-10.* The program gives the roots of the characteristic equation and the modal matrix.

4-39 Principal *modes.* The equations of motion $M\{\ddot{x}\}+K\{x\}=\{0\}$ can be uncoupled and expressed in terms of the **principal coordinates** as shown in Eq. **(4-41).** Assuming appropriate initial conditions for the systems in Prob. **4-35, (a)** write a program to uncouple the equations, (b) compute the vibrations in the original coordinates and the principal coordinates, and (c) plot the results for $t \simeq 1.5\tau$, where τ is the period of the first mode.

4-40 Modal analysis. Use the program TRESPUND in Fig. **9-8(*a*)** to find the transient response of the **system** in Fig. 4-10 for $0 \leq t \leq 1$. Choose the appropriate initial conditions and consider the problem in three parts as follows:

(a) $\{F(t)\}=\{0\}, \{x(0)\} \neq \{0\}$, and $\{\dot{x}(0)\} \neq \{0\}$.

(b) $\{F(t)\} \neq \{0\}, \{x(0)\}=\{0\}$, and $\{\dot{x}(0)\}=\{0\}$.

(c) $\{F(t)\} \neq \{0\}, \{x(0)\} \neq \{0\}$, and $\{\dot{x}(0)\} \neq \{0\}$.

Verify from the computer print-out that the value of $\{x(t)\}$ and $\{\dot{x}(t)\}$ from part c is 'the sum of that of parts a and b.

4-41 Transient response plot.

(a) Modify the program TRESPUND in Fig. **9-8(*a*)** such that the values of the displacement $\{x(t)\}$ are stored in one file and the velocity $\{\dot{x}(t)\}$ in another.

(b) Execute the program for the system in Fig. 4-10 and use the program **PLOTFILE** in Fig. **9-5(*a*)** to plot the results.

4-42 ***Dynamic*** absorber. The undamped dynamic absorber shown in Fig. 4-16 was analyzed in Example 12. (a) Write a program to implement Eq. (4-64) for the harmonic response and store the results in a data file. **(b)** Use the program **PLOTFILE** in Fig. **9-5(*a*)** to plot the results as illustrated in Fig. 4-17.

4-43 Dynamic absorber with damping was described in Example 13. Assume $k_1/m_1=k_2/m_2$ as for undamped absorbers. Let $m_2/m_1=0.3$. Select values for k_1, k_2, m_1, m_2, and five values for the damping coefficient c. Write a program to store the data in a file for (a) amplitude X_1 versus frequency, with c as a **parameter,** and (b) amplitude X_2 versus frequency with c as a parameter. Use the program **PLOTFILE** in Fig. **9-5(*a*)** to plot the results.

4-44 Repeat Prob. 4-43, but for the optimum condition $k_1/k_2=m_1m_2/(m_1+m_2)^2$ as shown in Prob. 4-27.

445 Vibration isolation. Vibration isolation for the system shown in Fig. 4-20 was discussed in Ekample 14. Let $m_1=180$ kg, $k_1=162$ **kN/m,** $m_2=4m_1$, $k_2=k_1$, $c_2=c_1$, and the mean excitation frequency f = **10** Hz. Assume $1.0 \leq c_1 \leq 4.0$ **kN·s/m.** Calculate the amplitudes X_1, X_2, and the force transmitted F_T to the foundation with c_1 as a parameter for $5 \leq f \leq 15$ Hz. (a) Write a program to store the calculated values of X_1 versus f in a data **file,** X_2 versus f in a second file, and F_T versus f in a third file. **(b)** Use the program **PLOTFILE** in Fig. **9-5(*a*)** to plot the results.

4-46 Frequency response. Consider an n-degree-of-freedom system with viscous damping

$$M\{\ddot{x}\}+C\{\dot{x}\}+K\{x\}=\{Q(t)\}$$

where M is the mass matrix, C the damping matrix, K the stiffness matrix, and $\{Q(t)\}$ the excitation vector. Assume each element of $\{Q(t)\}$ is harmonic and of the same frequency ω as discussed in **Sec.** 4-8. (a) Write a program for $n \leq 10$ to store the information of the amplitude versus ω and phase angles versus ω of the response in separate data files. (b) Use the program PLOTFILE in Fig. 9-5(a) to plot the results. For purpose of illustration, assume $0.1 \leq \omega \leq 2.0$ **rad/s** and the equations in the SI units are

$$\begin{bmatrix} 10 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}+\begin{bmatrix} 13 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 10 \sin \omega t \\ 0 \end{bmatrix}$$